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COMPLETELY INTEGRABLE SYSTEMS ON HAMILTONIAN MECHANICS

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Abstract

During the last two centuries, the study of mechanics has enjoyed a remarkable evolution, in parallel with one of its main mathematical tools: symplectic geometry. In this text, some of the most important notions have been gathered for the understanding of the Liouville-Arnold Theorem on completely integrable systems. The final goal of this project is to give a new approach to this fundamental result; thus the theory presented is appropriately nourished with humble examples to be analyzed. During the work previous to the final composition, several sources about both the main and many neighboring topics had to be studied. The tools given here can bring interested readers to the further study of gigantic problems such as the restricted three body problem, perturbation theory, and infinite dimensional integrable systems.

Resum

En el curs dels últims dos segles, l'estudi de la mecànica ha gaudit d'una notable evolució, així com ho ha fet una de les seves eines matemàtiques principals: la geometria simplèctica. En aquest text, s'han recollit algunes de les nocions més importants en l'enteniment del Teorema de Liouville-Arnold sobre sistemes completament integrables. L'objectiu últim d'aquest projecte és el d'acostar-se d'una nova manera a aquest resultat fonamental; és per això que la teoria presentada s'ha nodrit adequadament amb l'anàlisi d'alguns exemples humils. Durant la tasca prèvia a la composició definitiva es van haver d'estudiar diverses fonts, tant sobre el tema principal com sobre branques veïnes. Les eines que aquí es donen poden duur lectors interessats a l'estudi de problemes gegantins tals com el problema de tres cossos restringit, la teoria perturbacional i els sistemes integrables de dimensió infinita.

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Content

Preface

Many different mathematical methods and concepts are used in classical mechanics: differential equations and phase flows, smooth mappings and manifolds, Lie groups and Lie algebras, symplectic geometry and ergodic theory. Many modern mathematical theories arose from problems in mechanics and only later acquired that axiomatic-abstract form which makes them so hard to study.

The goal of this text is to provide the general public with a new approach to an all-time classic, hopefully in a way that one could find illustrating. By no means, though, it is wanted to suggest that genuine solutions, a new formalism or similar breakthroughs are proposed among these pages. *Au contraire*, the author's sole desire is to gather a set of notions that allow us to grasp the full meaning of the Liouville-Arnold Theorem. It is the specific collection of definitions and results contained under this preface that makes the work unique and therefore, in some sense, new.

These notes contain an introduction to symplectic geometry, followed by the basic notions of the hamiltonian formalism in classical mechanics. As the central part of the exposition, there stand the theory by Liouville and Arnold about completely integrable systems, their theorem and their action-angle variables (see **Theorem 4.3**). Some chapters bearing selected examples will be also found among their more theoretical counterparts. These examples will swarm around two names: Lagrange's spinning top and the restricted three body problem.

Within the mark of symplectic geometry, there is one main result we want to be focused onto. Darboux's theorem represents the first solid pillar from where to start building. It states that locally, symplectic manifolds with the same dimension are distinguishable from one another (see **Theorem 1.11**). This result opens a door to searching for systems on which the expressions adopt an arbitrarily comfortable shape.

The notion of Poisson brackets (related to the widely known Lie brackets) quickly allows us to state Noether's theorem (see **Theorem 2.14**). This theorem represents another one of the pillars that hold the whole structure. Physically speaking, it claims that in the motion of a system, for every preserved quantity[2] along the motion, one can find a symmetry[3]. Considering this result together with Darboux's, a new major goal presents itself: finding as many conserved quantities as one can, in order to get the easiest possible form for the equations of motion.

Steiner and Euler's (amongst others) work on rigid bodies brought the development of the inertia tensor. The second one did a particularly exploitable job finding privileged directions while studying the motion of such systems (see, for example **Theorem 3.13** or **Theorem 3.15**). The process followed with the definition of the inertia ellipsoid, an imaginary body defined from the principal axes that completely determines the evolution of the real object it represents. The existence of the inertia ellipsoid generalizes the study of rigid bodies, since a particular ellipsoid is associated to every system. Therefore, two systems with the same inertia ellipsoid will behave identically under the same conditions and force fields.

The Liouville-Arnold Theorem is the core result we want to describe. After appropriately defining what to understand as a first integral of a motion, some very strong assertions are manifested. Loosely speaking, it says that n first integrals are enough to completely solve the $2n$ equations of motion the hamiltonian formalism renders for a specific system. Furthermore, it proves the existence of what we have been introducing: a system of coordinates with which the differential equations system adopts the easiest possible form.

In order to capture the magnitude of the results we are going to discuss, let us have a brief

[2] A quantity remains constant on the whole of the orbit of motion, also named first integrals of the system.

[3] A parameter that turns out being zero in the mathematical expressions

chronological review. The Lagrangian point of view (1788) allowed us to solve completely a series of important mechanical problems, including problems in the theory of small oscillations and in the dynamics of a rigid body (see the introduction to Chapter 3). Making use of the Legendre transform, Hamilton proposed his own reformulation of classical mechanics (see Equation (2.1)) half a century afterwards (1833). It was the use of Hamiltonian mechanics that gave their title to a remarkable portion of today's *solvable* problems.

Both formulations had an incredibly important common point: for the first time, they provided the user with a system that could be formulated under virtually any coordinate system. In the end, the equations of motion of a mechanical problem come to us in the form of a set of differential equations. The Lagrange formalism associates a set of n second-order differential equations to a problem with n degrees of freedom. The Hamiltonian approach gives in turn a system of $2n$ first-order differential equations, much easier to solve.

Later on, the fruitfully broad career of Liouville meant great leaps on this and other directions, one of them consisting on defining the notion of integrable system. This is, of course, the one we are most interested in. During the last century, it was Arnold (preceded by his mentor Kolmogorov) who brought our understanding of such topics some steps beyond.

In 1954 Kolmogorov proved the existence of quasi-periodic solutions in perturbations of integrable Hamiltonian systems. That problem, motivated by the question of the stability of the solar system, had vexed mathematicians for almost a century. Arnold developed a different approach that yielded deeper insights, and, at around the same time, Moser enlarged the class of systems that could be treated. The resulting collection of mathematical theorems and techniques is now known as the Kolmogorov-Arnold-Moser (KAM) theory.

These are the historical milestones that motivate the developing of the present work. These might as well be some of the reasons why symplectic geometry underwent a golden era during the last century, since many have trusted Arnold's dictum saying that *"Hamiltonian mechanics cannot be understood without differential forms."*

A clear interest of the study of these topics is, following Kolmogorov's concerns, being able to find solid statements on the topic of perturbed systems within the restricted three body problem.

We would like to end this introduction with a quick scan of some keywords one can find in the following chapters.

Chapter 1. establishes the ground common knowledge on symplectic geometry: A symplectic form is a 2-form satisfying an algebraic condition -nondegeneracy- and an analytical condition -closedness. A symplectic manifold is a manifold equipped with a symplectic form. Symplectic geometry is the geometry of symplectic manifolds. Symplectic manifolds are necessarily even-dimensional. The closedness condition is a natural differential equation, which forces all symplectic manifolds to being locally indistinguishable (Darboux's Theorem). Equivalence between symplectic manifolds is expressed by a *symplectomorphism*. Physical transformations, the switch map and Hamiltonian transformations are given as examples of symplectomorphisms.

Chapter 2. has a similar purpose, yet now describing hamiltonian mechanics: Hamilton's equations provide $2n$ first-order differential equations describing a physical system. The same equations implicitly define the Hamiltonian vector space, with its associated phase flow. Lie derivatives, Lie brackets and Lie algebras provide a base to build the Poisson bracket. With the Poisson bracket one can already state and prove Noether's Theorem. The proof of Darboux's Theorem is also found here.

Chapter 3. presents the study of rigid body systems. As an introduction, a *crash-course* on Lagrangian formalism is included at the beginning of this chapter. A rigid body is a system of point masses where the distance between points is constant. Conservation laws are formulated here, and basic guidelines are given on how to find preserved quantities. The inertia operator works as an introduction to the consideration of the inertial reference frame *versus* the center of mass reference frame. Principal axes are here first considered and their natural next step: the inertia ellipsoid. Here, there are enough tools to consider the study of spinning tops. A Lagrangian top is a symmetric rigid body with a fixed stationary point O whose inertia ellipsoid at O is an ellipsoid of revolution and whose center of gravity lies on the axis of symmetry. The calculation of the lagrangian function opens the window into investigating its motion. Nutation is a periodic change on the top's inclination. Precession is the azimuthal sway motion of the top. The complete motion is composed by rotation among the symmetry axis, nutation and precession.

Chapter 4. is the central piece of the present exposition. The notion of first integral is defined and the Liouville-Arnold theorem on action-angle variables is stated. The proof is composed of several traces spread among the whole rest of the chapter. The second statement of the theorem presents a diffeomorphism between the given manifold and an n-dimensional invariant torus. Several lines are devoted to discrete subgroups, related to the coordinates mounted on the torus. To complete the chapter, there is a description of the proposed action-angle variables. Here one can see the enhancement these bring to the treatment of several problems. Finally one finds an outlining of a method for building action-angle variables in general cases.

Chapter 5. is the preamble for Chapter 6. Here the Kepler problem is considered. The Kepler problem is what we could understand under the name of the two body problem. First there is an exposition of the physics of central field problems. Central fields generally conserve angular momentum. The Kepler problem comes as a special case setting the central potential as Newton's gravity force. The Runge-Lenz vector is introduced as the second integral invariant of the problem. At this point the solution of Kepler's problem is presented. The rest of the chapter focuses on the study of the planar Kepler problem. The planar problem only comes as a special case of all the previous methods.

Chapter 6. wants to bring the reader closer to the study of the restricted three body problem. The restricted three body problem involves the motion of a massless body under the gravitational attraction of two massive bodies, which in turn attract one another as well. The restricted three body problem is first considered on an inertial frame. The exposition naturally leads then to the circular restricted three body problem. Next, there appear some required notions on time dependent transformations. The chapter ends with several computations and insights given on the circular problem on a rotating frame.

The images were taken from Arnold [2].

Acknowledgements

On a closing note, the author would like to thank Chris Wendl for unveiling such an amazing field of mathematical physics (or physical mathematics) on the first place, and for suggesting the development of this particular Theorem, which will definitely shift the way physics students look at Classical Mechanics. This work would have never existed today without him.

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Chapter 1

Symplectic Geometry

A symplectic structure on a manifold is a closed nondegenerate differential 2-form. The phase space of a mechanical system has a natural symplectic structure.

As a curiosity, note that two centuries ago the name *symplectic geometry* did not exist. If you consult a major English dictionary, you are likely to find that *symplectic* is the name for a bone in a fish's head. However, the word *symplectic* in mathematics was coined by Weyl who substituted the Latin root in *complex* by the corresponding Greek root, in order to label the symplectic group. Weyl thus avoided that this group connoted the complex numbers, and also spared us from much confusion that would have arisen, had the name remained the former one in honor of Abel: *abelian linear group*.

1.1 Symplectic Manifolds

Definition 1.1. *A **symplectic manifold** is a tuple (M, ω) where M is a differentiable manifold and $\omega \in \Omega^2(M)$ is a two-form on M satisfying two properties: it is closed, and it is non-degenerate.*

$$d\omega = 0 \quad \text{and } \forall \xi \neq 0,\ \exists \eta : \omega(\xi, \eta) \neq 0, \quad \text{where } \xi, \eta \in TM_x, \text{ for every } x \in M.$$

*The two-form ω is called the **symplectic structure** on M.*

The assumption that ω is non-degenerate immediately implies that a symplectic manifold is even dimensional. In other words an odd dimensional manifold never admits a symplectic structure.

Definition 1.2. *A symplectic vector space (M, ω) has a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ satisfying*

$$\omega(e_i, f_i) = \delta_{ij} \quad \text{and} \quad \omega(e_i, e_j) = 0 = \omega(f_i, f_j).$$

*Such a basis is called a **symplectic basis** of (M, ω).*

In mechanics, the commonly used symplectic basis of a $2n$-dimensional symplectic manifold is called the set of canonical coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$[1]. The q components are referred to as **spatial coordinates**, whereas the p components are referred to as the **momenta**.

The archetypical example of a symplectic manifold is the cotangent bundle of a smooth manifold. Assume that N is a manifold, by physicists also referred to as the **configuration space**,

[1] Note that sometimes we will alter the order of the components, so the base (p, q) will also be called canonical in some cases, due to different traditions in math and physics.

where the q's live. The **phase space** is the cotangent bundle T^*N, the subspace of both q's and p's. The cotangent bundle comes endowed with a canonical one-form called the **Liouville one-form**. It is defined as follows. Abbreviate by $\pi : T^*N \to N$ the footpoint projection. If $e \in T^*N$ and $\xi \in T_eT^*N$, the tangent space of T^*N at e, the differential of the footpoint projection at e is a linear map

$$d\pi(e) : T_eT^*N \to T_{\pi(e)}N.$$

Interpreting e as a vector in $T^*_{\pi(e)}N$ the dual space of $T_{\pi(e)}N$, we can pair it with the vector $d\pi(e)\xi \in T_{\pi(e)}N$ and define

$$\lambda_e(\xi) := e(d\pi(e)\xi).$$

In canonical coordinates $(p, q) = (q_1, \dots, q_n, p_1, \dots p_n)$ of T^*N, where $n = \dim N$, the Liouville one-form becomes

$$\lambda(q, p) = \sum_{i=1}^{n} p_i dq_i.$$

Definition 1.3. *The* ***canonical symplectic (two-)form*** *on T^*N is the exterior derivative of the Liouville one-form*

$$\omega = d\lambda.$$

Clearly ω is closed because, according to the properties of exterior differentiation $d\omega = d(d\lambda) = (d \cdot d)\lambda \equiv 0$. In canonical coordinates it has the form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

1.2 Symplectomorphisms

Definition 1.4. *Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a* ***symplectomorphism*** *if $\varphi^*\omega_2 = \omega_1$. If a symplectomorphism exists between (M_1, ω_1) and (M_2, ω_2), then these are said to be* ***symplectomorphic****.*

Let us now consider different symplectomorphism examples.

1.2.1 Physical transformations

Suppose that N_1 and N_2 are manifolds and $\phi : N_1 \to N_2$ is a diffeomorphism, for example a change of coordinates of the configuration space. If $x \in N_1$ the differential

$$d\phi(x) : T_xN_1 \to T_{\phi(x)}N_2$$

is a vector space isomorphism. Dualizing we get yet another vector space isomorphism

$$d\phi(x)^* : T^*_{\phi(x)}N_2 \to T^*_xN_1.$$

We now define

$$d_*\phi : T^*N_1 \to T^*N_2$$

as follows. If $\pi_1 : T^*N_1 \to N_1$ is the footpoint projection and $e \in T^*N_1$, then

$$d_*\phi(e) := (d\phi(\pi_1(e))^*)^{-1}e. \tag{1.1}$$

If λ_1 is the Liouville one-form on T^*N_1 and λ_2 is the Liouville one-form on T^*N_2 one checks that

$$(d_*\phi)^*\lambda_2 = \lambda_1.$$

Because exterior derivative commutes with pullback we obtain

$$(d_*\phi)^*\omega_2 = (d_*\phi)^*d\lambda_2 = d(d_*\phi)^*\lambda_2 = d\lambda_1 = \omega_1 \tag{1.2}$$

which shows that $d_*\phi$ is a symplectomorphism.

Definition 1.5. *We will say a symplectomorphism is* ***exact*** *if it preserves the primitives of the symplectic forms.*

Equation 1.2 might be rephrased by saying that $d_*\phi$ is an exact symplectomorphism. The notion of exact symplectomorphism in a general symplectic manifold however does not make sense, since usually the symplectic form ω induces a non-vanishing class $[\omega] \in H^2_{dR}(M)$, the second de Rham cohomology group of M. In particular, ω cannot be exact.

1.2.2 The switch map

The second example of a symplectomorphism is the switch map

$$\sigma : T^*\mathbb{R}^n \to T^*\mathbb{R}^n.$$

Namely, if we identify the cotangent bundle $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$ with global canonical coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ and symplectic form $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ then the switch map is given by

$$\sigma(q, p) = (-p, q)$$

which is actually a linear symplectomorphism on $\mathbb{R}^{2n}$. Note that the switch map is not a physical transformation. The switch map interchanges the roles of the momenta and the positions. Note that in order to define the switch map it is important to have global coordinates on the configuration space. There is no way to define the switch map on the cotangent bundle T^*N of a general manifold N.

1.2.3 Hamiltonian transformations

The third example of symplectomorphisms we discuss are Hamiltonian transformations. Suppose that (M, ω) is a symplectic manifold and $H \in C^\infty(M)$.

Definition 1.6. *Smooth functions on a symplectic manifold are referred to by physicists as* ***hamiltonians*** *or* ***hamiltonian functions****.*

The interesting point about hamiltonians is that we can associate to them a vector field $X_H \in \Gamma(TM)$ which is implicitly defined by the condition

$$dH = \omega(\cdot, X_H).$$

Note that the assumption that the symplectic form is non-degenerate guarantees that X_H is well defined.

Definition 1.7. *The vector field X_H is called* ***Hamiltonian vector field****.*

Let us assume for simplicity in the following that M is closed. Under this assumption the flow of the Hamiltonian vector field exists for all times, i.e., we get a smooth family of diffeomorphisms

$$\phi_H^t : M \to M, \quad t \in \mathbb{R}$$

defined by the conditions

$$\phi_H^0 = \mathrm{id}_M, \quad \frac{d}{dt}\phi_H^t = X_H(\phi_H^t(x)),\, t \in \mathbb{R},\, x \in M.$$

An important property of the Hamiltonian flow is that the Hamiltonian H is preserved under it. In many cases, the hamiltonian function in mechanical systems coincides with the mechanical energy of the system, in those cases we can say the energy is conserved:

Theorem 1.8. ***(Preservation of energy)*** *For $x \in M$ it holds that $H(\phi^t(x))$ is constant, i.e., independent of $t \in \mathbb{R}$.*

Proof. Differentiating we obtain

$$\begin{aligned}
\frac{d}{dt}H(\phi_H^t(x)) &= dH(\phi_H^t(x))\frac{d}{dt}\phi_H^t(x) \\
&= dH(\phi_H^t(x))X_H(\phi_H^t(x)) \\
&= \omega(X_H, X_H)(\phi_H^t(x)) \\
&= 0
\end{aligned}$$

where the last equality follows from antisymmetry of the two-form ω. □

The next theorem tells us that the diffeomorphisms ϕ_H^t are symplectomorphisms. The intuition from physics might be that a Hamiltonian system has *no friction*.

Theorem 1.9. *For every $t \in \mathbb{R}$ it holds that $(\phi_H^t)^*\omega = \omega$*

Proof. Note that

$$\frac{d}{dt}(\phi_H^t)^*\omega = (\phi_H^t)^*\mathcal{L}_{X_H}\omega$$

where $\mathcal{L}_{X_H}\omega$ is the Lie derivative of the symplectic form with respect to the Hamiltonian vector field. Using Cartan's formula we obtain by taking advantage of the assumption that ω is closed and the definition of X_H

$$\mathcal{L}_{X_H}\omega = \imath_{X_H}d\omega + d\imath_{X_H}\omega = -d^2H = 0.$$

This proves the theorem. □

1.3 Darboux's Theorem

The relation of being symplectomorphic is clearly an equivalence relation in the set of all even-dimensional vector spaces. Therefore, every $2n$-dimensional symplectic vector space (M, ω) can be seen as symplectomorphic to a certain prototype $(\mathbb{R}^{2n}, \omega_0)$; a choice of a symplectic basis for (M, ω) yields a symplectomorphism to $(\mathbb{R}^{2n}, \Omega_0)$. Hence, non negative even integers classify equivalence classes for the relation of being symplectomorphic.

Recall that, by definition of pullback, at tangent vectors $u, v \in T_pM_1$, we have $(\varphi^*\omega_2)_p(u,v) = (\omega_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v))$.

We would like to classify symplectic manifolds up to symplectomorphism. The Darboux theorem takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$, any $2n$-dimensional symplectic manifold looks locally like $(\mathbb{R}^{2n}, \omega_0)$. More precisely, any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Recall that the definition of a manifold includes a compatibility condition for the charts of an atlas. This is a condition on the maps $\varphi_i^{-1}\varphi_j$ going from one chart to another. The maps $\varphi_i^{-1}\varphi_j$ are maps of a region of coordinate space.

Definition 1.10. *An atlas of a manifold M^{2n} is called **symplectic** if the standard symplectic structure $\omega = dp \wedge dq$ is introduced into the coordinate space $\mathbb{R}^{2n} = \{(p,q)\}$, and the transfer from one chart to another is realized by a **canonical** (i.e., ω-preserving) transformation $\varphi_i^{-1}\varphi_j$.*

It can be seen that every symplectic atlas defines a symplectic structure on M^{2n} and, conversely, every symplectic manifold has a symplectic atlas. These follow from the following theorem.

Theorem 1.11. ***(Darboux)*** *Let ω be a closed nondegenerate differential 2-form in a neighborhood of a point x in the space $\mathbb{R}^{2n}$. Then in some neighborhood of x one can choose a coordinate system $(p_1, \dots, p_n; q_1, \dots, q_n)$ such that the form has the standard form:*

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

This theorem allows us to extend to all symplectic manifolds any assertion of a local character which is invariant with respect to *canonical transformations*[2] and is proven for the standard phase space ($\mathbb{R}^{2n}$, $\omega = dp \wedge dq$).

In order to give a complete proof of this Theorem we still need to give some analytical tools, such as the notion of Poisson brackets. It will be proved in section 2.2.1.

Further reading

Some wider base of notions might be interesting to compliment the ones given here, for instance, chapters 7 and 8 of Arnold [2] or the first sections of Bolsinov and Fomenko [5] and Cannas da Silva [6].

[2]Traditionally used name for symplectomorphisms in the study of Hamiltonian mechanics from a physicist viewpoint.

Chapter 2

Hamiltonian Mechanics

Hamiltonian mechanics is geometry in phase space. The group of symplectic diffeomorphisms acts on phase space. The basic concepts and theorems of hamiltonian mechanics (even when formulated in terms of local symplectic coordinates) are invariant under this group.

A hamiltonian mechanical system is given by an even-dimensional manifold, a symplectic structure on it and a function on it. Every one-parameter group of symplectic diffeomorphisms of the phase space preserving the hamiltonian function is associated to a first integral of the equations of motion.

The hamiltonian point of view allows us to solve completely a series of mechanical problems which do not yield solutions by other means. The hamiltonian point of view has even greater value for the approximate methods of perturbation theory, for understanding the general character of motion in complicated mechanical systems and in connection with other areas of mathematical physics.

2.1 Classical Mechanics

As it has been already mentioned, the revolution on the study of Classical Mechanics came first by the hand of Lagrange. Taking advantage of variational principles he provided a set of second order differential equations which, when applied on the function named after him (the lagrangian function), gave the properties of the motion of a mechanical system. Together with these mathematical objects, there was a whole apparatus to hold them. The true genius of this new approach to a centuries-long problem lay indeed on this apparatus. Following his path, one could attack a problem without having to carry around annoying terms on differential equations that arose from systems not being naturally well described by cartesian coordinates[1]. The definition of *generalized coordinates* brought to the use of *generalized momenta* and *generalized forces*, with which visual inspection of the equations was often enough to reach an unheard of amount of insight about the studied system.

The second big breakthrough following Lagrange eventually entitled this whole branch of analytical mechanics: hamilton's equations. Hamilton, making use of the Legendre transform could bring the Euler-Lagrange equations yet one step beyond. The use of the contemporary mathematical tools allowed him to go from a system of n second-order differential equations (Lagrange's equations) to a system of $2n$ first-order differential equations, much friendlier to

[1]Note the subjectivity of the term "well described"; well, in many cases it meant for a system to be solvable analytically or not.

disentangle, of course. With the new techniques yet another significant amount of problems and systems were disclosed, and reasonable explanations to truly complex systems were being given for the first time. Here is where we start our exposition, with the definition of the set of equations Hamilton gave us to describe the phzsical world.

Consider the euclidean space $\mathbb{R}^{2n}$ with canonical coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and $\omega_0 = \sum dq_j \wedge dp_j$ [2]. The curve $\rho_t = (q(t), p(t))$ is an integral curve for X_H if

$$\begin{cases} \frac{dq_i}{dt}(t) = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt}(t) = -\frac{\partial H}{\partial q_i} \end{cases} \quad \textbf{(Hamilton equations)} \tag{2.1}$$

Indeed, let $X_H = \sum_{i=1}^n (\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i})$. Then,

$$\begin{aligned} \imath_{X_H}\omega &= \sum_{j=1}^n \imath_{X_H}(dq_j \wedge dp_j) = \sum_{j=1}^n \left[(\imath_{X_H} dq_j) \wedge dp_j - dq_j \wedge (\imath_{X_H} dp_j) \right] \\ &= \sum_{j=1}^n \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) = dH. \end{aligned}$$

Remark 2.1. The gradient vector field of H relative to the euclidean metric is

$$\nabla H := \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i}\frac{\partial}{\partial q_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial p_i} \right).$$

The case where $n = 3$ has a simple physical illustration. Newton's second law states that a particle of mass m moving in **configuration space** $\mathbb{R}^3$ with coordinates $q = (q_1, q_2, q_3)$ under a potential $V(q)$ moves along a curve $q(t)$ such that

$$m\frac{d^2q}{dt^2} = -\nabla V(q).$$

Introduce the **momenta** $p_i = m\frac{dq_i}{dt}$ for $i = 1, 2, 3$, and **energy** function $H(p, q) = \frac{1}{2m}|p|^2 + V(q)$. Let $\mathbb{R}^6 = T^*\mathbb{R}^3$ be the corresponding **phase space**, with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$. Newton's second law in $\mathbb{R}^3$ is equivalent to the Hamilton equations in $\mathbb{R}^6$:

$$\begin{cases} \frac{dq_i}{dt} = \frac{1}{m}p_i = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = m\frac{d^2q_i}{dt^2} = -\frac{\partial V}{\partial q_i} = -\frac{\partial H}{\partial q_i}. \end{cases}$$

The energy H is conserved along the motion.

2.2 Poisson brackets and Noether's theorem

Vector fields are differential operators on functions: if X is a vector field and $f \in C^\infty(M)$, df being the corresponding 1-form, then

$$X \cdot f := df(X) = \mathcal{L}_X f.$$

Where $\mathcal{L}$ denotes the **Lie derivative**. In order to simplify differential equations, it is important to identify *preserved quantities*, also called integrals.

[2]One can define canonical coordinates as (q, p) or (p, q), traditionally both options have been used. This choice determines ω_0; depending on the notation, a minus sign will need to be added in front of the 2-form.

Definition 2.2. *If X is a vector field on a manifold M, then we call L an **integral** of X if $X(L) = 0$.*

Given two vector fields X, Y, there is a unique vector field W such that

$$\mathcal{L}_W f = \mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f).$$

The vector field W is called the **Lie bracket** of the vector fields X and Y and denoted $W = [X, Y]$, since $\mathcal{L}_W = [\mathcal{L}_X, \mathcal{L}_Y]$ is the commutator.

Proposition 2.3. *If X and Y are symplectic vector fields on a symplectic manifold (M, ω), then $[X, Y]$ is hamiltonian with hamiltonian function $\omega(Y, X)$.*

Proof.

$$\begin{aligned}
\imath_{[X,Y]}\omega &= \mathcal{L}_X \imath_Y \omega - \imath_Y \mathcal{L}_X \omega \\
&= d\imath_X \imath_Y \omega + \imath_X \underbrace{d\imath_Y \omega}_{0} - \imath_Y \underbrace{d\imath_X \omega}_{0} - \imath_Y \imath_X \underbrace{d\omega}_{0} \\
&= d(\omega(Y, X)).
\end{aligned}$$

□

Definition 2.4. *A (real) **Lie algebra** is a (real) vector space $\mathfrak{g}$ together with a **Lie bracket** $[\cdot, \cdot]$, i.e., a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:*

(a) $[x, y] = -[y, x], \quad \forall x, y \in \mathfrak{g},$ ***(antisymmetry)***

(b) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}.$ ***(Jacobi identity)***

Let us use the following terminology:

$$\begin{aligned}
\chi(M) &= \{\text{vector fields on } M\} \\
\chi^{\text{sympl}}(M) &= \{\text{symplectic vector fields on } M\} \\
\chi^{\text{ham}} &= \{\text{hamiltonian vector fields on } M\}.
\end{aligned}$$

Corollary 2.5. *The inclusions $\left(\chi^{ham}(M), [\cdot, \cdot]\right) \subseteq \left(\chi^{sympl}(M), [\cdot, \cdot]\right) \subseteq (\chi(M), [\cdot, \cdot])$ are inclusions of Lie algebras.*

Definition 2.6. *For a symplectic manifold (M, ω) we define the **Poisson bracket** of smooth functions F and G by*

$$\{F, G\} := \omega(X_F, X_G) = -dF(X_G) = -X_G(F) = X_F(G). \tag{2.2}$$

We see directly from the definition that the Poisson bracket describes the time-evolution of a function. Indeed, suppose that $\gamma(t)$ is a flow line of X_F. Then

$$\frac{dG \circ \gamma(t)}{dt} = X_F(G) = \{F, G\}.$$

From this, energy preservation (see **Theorem 1.8**) follows too because $\{H, H\} = 0$ (the Poisson bracket is alternating). Before we turn our attention to conserved quantities, we first need to establish some properties of the Poisson bracket.

Lemma 2.7. *Given smooth functions F, G on a symplectic manifold (M, ω), we have the following relation between the Lie bracket and the Poisson bracket,*

$$[X_F, X_G] = X_{\{F,G\}}$$

Depending on conventions, the same statement can be reached with a minus sign in one of sides.

Proof. We first rewrite the Lie bracket a bit:

$$[X_F, X_G] = \mathcal{L}_{X_F} X_G = \frac{d}{dt}\Big|_{t=0} Fl_t^{X_F *} X_G = \frac{d}{dt}\Big|_{t=0} X_{G \circ Fl_t^{X_F}}.$$

Now use this identity and the definition:

$$\begin{aligned} i_{[X_F, X_G]}\omega &= \frac{d}{dt}\Big|_{t=0} \omega(X_{G\circ Fl_t^{X_F}}, \cdot) \\ &= \frac{d}{dt}\Big|_{t=0} (-d(G \circ Fl_t^{X_F})) \\ &= -d\left(\frac{d}{dt}\Big|_{t=0} G \circ Fl_t^{X_F}\right) = -d(X_F(G)) = -d\{F, G\}. \end{aligned}$$

□

Theorem 2.8. *The bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity, i.e.,*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Definition 2.9. *A **Poisson algebra** $(\mathcal{P}, \{\cdot, \cdot\})$ is a commutative associative algebra $\mathcal{P}$ with a Lie bracket $\{\cdot, \cdot\}$ satisfying the **Leibniz rule**:*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

We conclude that, if (M, ω) is a symplectic manifold, then $(C^\infty(M), \{\cdot, \cdot\})$ is a Poisson algebra.

In a Darboux chart $(U, \omega = dp \wedge dq)$ that is, on canonical coordinates for (M, ω), the Poisson bracket can be written as

$$\{F, G\} = \sum_i \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}.$$

In a moment, we shall see that the Poisson bracket endows the space of smooth functions on M with a Lie algebra structure. We briefly recall the definition.

Proposition 2.10. *The pair $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra.*

Proof. We check the required properties. First of all, note that for $a \in \mathbb{R}$ and $F, G \in C^\infty(M)$, we have $X_{aF+G} = aX_F + X_G$, since the Hamiltonian vector field is the solution to a linear equation. Hence, $\{aF + G, H\} = \omega(X_{aF+G}, X_H) = a\omega(X_F, X_H) + \omega(X_G, X_H) = a\{F, H\} + \{G, H\}$. The same argument works for the second factor, so $\{\cdot, \cdot\}$ is bilinear. Also, $\{F, F\} = \omega(X_F, X_F) = 0$, so $\{\cdot, \cdot\}$ is alternating. Alternatively, we can also use Lemma 2.7. Finally, we check that the Jacobi identity holds by computing the individual terms:

$$\begin{aligned} \{F, \{G, H\}\} &= X_F(\{G, H\}) = X_F(X_G(H)) \\ \{G, \{H, F\}\} &= -X_G(\{F, H\}) = -X_G(X_F(H)) \\ \{H, \{F, G\}\} &= -X_{\{F,G\}}(H) = -[X_F, X_G](H) \end{aligned}$$

We have used Lemma 2.7 in the last step. Summing these terms shows that the Jacobi identity holds. □

Lemma 2.11. *The function G is an integral of X_F if and only if $\{F, G\} = 0$.*

Proof. The function G is an integral if and only if $X_F(G) = 0$. This holds if and only if $0 = -dF(X_G) = \omega(X_F, X_G) = \{F, G\}$. $\square$

Remark 2.12. By Lemma 2.7, $\{F, G\} = 0$ implies that $[X_F, X_G] = 0$. On the other hand, for G to be an integral of X_F, it is not enough to just have $[X_F, X_G] = 0$. Indeed, consider $(\mathbb{R}^2, \omega_0 = dp \wedge dq)$ with the Hamiltonians $F = p$ and $G = q$. Then $X_F = \partial_q$ and $X_G = -\partial_p$, so $[X_F, X_G] = 0$. However, G is linearly increasing under the flow of X_F, so G is not an integral of X_F. On the other hand the next Lemma shows that if M is closed such a phenomenon cannot happen.

Lemma 2.13. *Assume that (M, ω) is a closed symplectic manifold and $F, G \in C^\infty(M)$ are two smooth functions such that $[X_F, X_G] = 0$. Then $\{F, G\} = 0$*

Proof. Because the commutator of the two Hamiltonian vector fields vanishes, it follows from Lemma 2.7 that

$$X_{\{F,G\}} = [X_F, X_G] = 0.$$

We assume without loss of generality that M is connected (otherwise we treat each connected component of M separately). Therefore we conclude that

$$\{F, G\} = c$$

where $c \in \mathbb{R}$ is a constant. Pick $x \in M$. To study the behavior of G along the flow of X_F through x we differentiate

$$\frac{d}{dt} G(\phi_F^t(x)) = dG(\phi_F^t(x)) X_F(\phi_F^t(x)) = \{F, G\}(\phi_F^t(x)) = c.$$

We conclude that

$$G(\phi_F^t(x)) = G(x) + ct.$$

Because M is compact by assumption the function G is bounded and therefore

$$c = 0.$$

This proves that F and G Poisson commute. $\square$

We are now in position to prove Noether's theorem. This is surely not its most common formulation, but it reveals a significant amount of information.

Theorem 2.14. ***(Noether)*** *Assume that (M, ω) is a closed symplectic manifold and $F, G \in C^\infty(M, \mathbb{R})$. Then the following are equivalent.*

1. *G is an integral for the flow of F, i.e., $G(\phi_F^t(x))$ is independent of t for every $x \in M$.*
2. *The flow of G is a symmetry for F, i.e., $F(\phi_G^t(x))$ is independent of t for every $x \in M$.*
3. *F and G Poisson commute, i.e., $\{F, G\} = 0$.*
4. *The flows of X_F and X_G commute, i.e., $[X_F, X_G] = 0$.*

Proof. That assertion 1 is equivalent to assertion 3 is the content of Lemma 2.11. Because the Poisson bracket is antisymmetric, the vanishing of $\{F, G\}$ is equivalent to the vanishing of $\{G, F\}$ and therefore assertion 3 is equivalent as well to assertion 2. That assertion 3 is equivalent to assertion 4 follows from Lemma 2.7 and Lemma 2.13. $\square$

2.2.1 Proof of Darboux's Theorem

Now we are ready to face the task of proving Theorem 1.11

Proof. Begin with the construction of the coordinates p_1 and q_1.

For the first coordinate p_1 we take a non-constant linear function (we could have taken any differentiable function whose differential was not zero at the point x). For simplicity we will assume that $p_1(x) = 0$.

It is clear that p_1 is indeed a hamiltonian. Let therefore $P_1 = X_{p_1}$ denote the hamiltonian field corresponding to the function p_1. Note that $P_1(x) \neq 0$; therefore, we can draw a hyperplane N^{2n-1} through the point x which does not contain the vector $P_1(x)$ (we could have taken any surface transverse to $P_1(x)$ as N^{2n-1}).

Consider the hamiltonian flow P_1^t with hamiltonian function p_1. We consider the time t necessary to go from N to the point $z = P_1^t(y)$ $(y \in N)$ under the action of P_1^t as a function of the point z. By the usual theorems in the theory of ordinary differential equations, this function is defined and differentiable in a neighborhood of the point $x \in \mathbb{R}^{2n}$. Denote it by q_1. Note that $q_1 = 0$ on N and that the derivative of q_1 in the direction of the field P_1 is equal to 1. Thus the Poisson bracket of the functions q_1 and p_1 we constructed is equal to 1:

$$\{q_1, p_1\} \equiv 1.$$

Now, let's consider the construction of symplectic coordinates by induction on n.

If $n = 1$, the construction is finished. Let $n > 1$. We will assume that Darboux's theorem is already proved for $\mathbb{R}^{2n-2}$. Consider the set M given by the equations $p_1 = q_1 = 0$. The differentials dp_1 and dq_1 are linearly independent at x since $\omega(X_{p_1}, X_{q_1}) = \{p_1, q_1\} \equiv 1$. Thus, by the implicit function theorem, the set M is a manifold of dimension $2n-2$ in a neighborhood of x; we will denote it by M^{2n-2}.

Lemma 2.15. *The symplectic structure ω on $\mathbb{R}^{2n}$ induces a symplectic structure on some neighborhood of the point x on M^{2n-2}.*

Proof. For the proof we need only the nondegeneracy of ω on TM_x. Consider the symplectic vector space $T\mathbb{R}_x^{2n}$. The vectors $P_1(x)$ and $Q_1(x)$ of the hamiltonian vector fields with hamiltonian functions p_1 and q_1 belong to $T\mathbb{R}_x^{2n}$. Let $\xi \in TM_x$. The derivatives of p_1 and q_1 in the direction ξ are equal to zero. This means that $dp_1(\xi) = \omega(\xi, P_1) = 0$ and $dq_1(\xi) = \omega(\xi, Q_1) = 0$. Thus TM_x is the skew-orthogonal complement to $P_1(x)$, $Q_1(x)$. Then, because of it being a symplectic basis, the form ω on TM_x is nondegenerate. □

By the induction hypothesis there are symplectic coordinates in a neighborhood of the point x on the symplectic manifold $(M^{2n-2}, \omega|_M)$. Denote them by p_i, q_i $(i = 2, \dots, n)$. We extend the functions $p_2, \dots, q_n$ to a neighborhood of x in $\mathbb{R}^{2n}$ in the following way. Every point y in a neighborhood of x in $\mathbb{R}^{2n}$ can be uniquely represented in the form $z = P_1^t Q_1^s w$, where $w \in M^{2n-2}$, and s and t are small numbers. We set the values of the coordinates $p_2, \dots, q_n$ at z equal to their values at the point w. The $2n$ functions $p_1, \dots, p_n, q_1, \dots, q_n$ thus constructed form a local coordinate system in a neighborhood of x in $\mathbb{R}^{2n}$.

Finally, we need to prove that the coordinates constructed are indeed symplectic.

Denote by P_i^t and Q_i^t $(i = 1, \dots, n)$ the hamiltonian flows with hamiltonian functions p_i and q_i, and by P_i and Q_i the corresponding vector fields. We will compute the Poisson brackets of the functions $p_1, \ldots, q_n$. We already saw that $\{q_1, p_1\} \equiv 1$. Therefore, the flows P_1^t and Q_1^t commute: $P_1^t Q_1^s = Q_1^s P_1^t$.

Recalling the definitions of $p_2, \ldots, q_n$ we see that each of these functions is invariant with respect to the flows P_1^t and Q_1^t. Thus the Poisson brackets of p_1 and q_1 with all $2n-2$ functions p_i, q_i $(i > 1)$ are equal to zero.

The map $P_1^t Q_1^s$ therefore commutes with all $2n-2$ flows P_i^t, Q_i^s $(i > 1)$. Consequently, it leaves each of the $2n-2$ vector fields P_i, Q_i $(i > 1)$ fixed. $P_1^t Q_1^s$ preserves the symplectic structure ω since the flows P_1^t and Q_1^s are hamiltonian; therefore, the values of the form ω on the vectors of any two of the $2n-2$ fields P_i, Q_i $(i > 1)$ are the same at the points $z = P_1^t Q_1^s\, w \in \mathbb{R}^{2n}$ and $w \in M^{2n-2}$. But these values are equal to the values of the Poisson brackets of the corresponding hamiltonian functions. Thus, the values of the Poisson bracket of any two of the $2n-2$ coordinates pi, q_i $(i > 1)$ at the points z and w are the same if $z = P_1^t Q_1^s\, w$.

The functions p_1 and q_1 are first integrals of each of the $2n-2$ flows P_i, Q_i $(i > 1)$., Therefore, each of the $2n-2$ fields P_i, Q_i is tangent to the level manifold $p_1 = q_1 = 0$. But his manifold is M^{2n-2}. Consequently, these fields are hamiltonian fields on the symplectic manifold $(M^{2n-2}, \omega|_M)$, and the corresponding hamiltonian functions are $p_i|_M$, $q_i|_M$ $(i > 1)$. Thus, in the whole space $(\mathbb{R}^{2n}, \omega)$, the Poisson bracket of any two of the $2n-2$ coordinates p_i, q_i $(i > 1)$ considered on M^{2n-2} is the same as the Poisson bracket of these coordinates in the symplectic space $(M^{2n-2}, \omega|_M)$.

But, by our induction hypothesis, the coordinates on M^{2n-2} are symplectic. Therefore, in the whole space $\mathbb{R}^{2n}$, the Poisson brackets of the constructed coordinates have the standard values

$$\{p_i, p_j\} \equiv \{p_i, q_j\} \equiv \{q_i, q_j\} \equiv 0 \text{ and } \{q_i, p_i\} \equiv 1.$$

The Poisson brackets of the coordinates p, q on $\mathbb{R}^{2n}$ have the same form if $\omega = \sum dp_i \wedge dq_i$. But a bilinear form ω is determined by its values on pairs of basis vectors. Therefore, the Poisson brackets of the coordinate functions determine the shape of ω uniquely. Thus

$$\omega = dp_1 \wedge pq_1 + \ldots + dp_n \wedge dq_n,$$

and Darboux's theorem is proved. □

Further reading

Many central topics of classical mechanics had to be left out of this brief introduction. For example one can read more on canonical transformations and generating functions in section 5.2 of Abraham and Marsden [1].

Also, several examples of Hamiltonian systems can be found everywhere, but keeping in mind that the goal of Frauenfelder and van Koert [8] is the study of the restricted three body problem, they present them in a much useful way for a greater understanding of subsequent problems.

Chapter 3

Rigid bodies

In this chapter we study in detail some very special mechanical problems. These problems are traditionally included in a course on classical mechanics, first because they were solved by Euler and Lagrange, and also because we live in three-dimensional euclidean space, so that most of the mechanical systems with a finite number of degrees of freedom which we are likely to encounter consist of rigid bodies.

For this specific chapter, recall some of the following notions, which belong the the Lagrangian formalism:

Definition 3.1. *The equation*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

is called the ***Euler-Lagrange equation***

Consider Newton's equations of dynamics

$$\frac{d}{dt}(m_i \dot{r}_i) + \frac{\partial U}{\partial r_i} = 0. \tag{3.1}$$

Theorem 3.2. ***(Hamilton's form of the principle of least motion)*** *Motions of the mechanical system 3.1 coincide with extremals of the functional*

$$\Phi(\gamma) = \int_{t_0}^{t_1} L\,dt,$$

where the ***Lagrangian function*** *or* ***lagrangian*** $L = T - U$ *is the difference between the kinetic and potential energy.*

It can be proved that *extremals* of the given *functional* are curves which fulfill the Euler-Lagrange equation.

Definition 3.3. *In mechanics, we use the following terminology:* $L(q, \dot{q}, t) = T - U$ *is the* ***Lagrange function*** *or* ***lagrangian****,* q_i *are the* ***generalized coordinates****,* $\dot{q}_i$ *are* ***generalized velocities****,* $\partial L/\partial \dot{q}_i = p_i$ *are* ***generalized momenta****,* $\partial L/\partial q_i$ *are* ***generalized forces****,* $\int_{t_0}^{t_1} L(q, \dot{q}, t)\,dt$ *is the* ***action****,* $(d(\partial L/\partial \dot{q}_i)/dt) - (\partial L/\partial q_i) = 0$ *are* ***Lagrange's equations****.*

In many cases the action $q(t)$ is not only an extremal but is also a minimum value of the action functional $\int_{t_0}^{t_1} L\,dt$.

3.1 Rigid bodies

3.1.1 The configuration manifold of a rigid body

Definition 3.4. *A **rigid body** is a system of point masses where the distance between points is constant:*

$$|x_i - x_j| = r_{ij} = const. \tag{3.2}$$

Theorem 3.5. *The configuration manifold of a rigid body is a six-dimensional manifold, namely, $\mathbb{R}^3 \times SO(3)$ (the direct product of a three-dimensional space $\mathbb{R}^3$ and the group $SO(3)$ of its rotations), as long as there are three points in the body not in a straight line.*

Proof. Let x_1, x_2 and x_3 be three points of the body which do not lie in a straight line. Consider the right-handed orthonormal frame whose first vector is in the direction of $x_2 - x_1$, and whose second one is on the x_3 side in the $x_1x_2x_3$-plane (Figure 3.1). It follows from the conditions $|x_i - x_j| = r_{ij}$ $(i = 1, 2, 3)$, that the positions of all the points of the body are uniquely determined by the positions of x_1, x_2 and x_3, which are given by the position of the frame. Finally, the space of frames in $\mathbb{R}^3$ is $\mathbb{R}^3 \times SO(3)$, since every frame is obtained from a fixed one by a rotation and a translation.[1] □

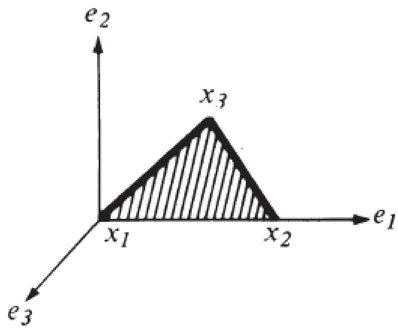

Figure 3.1: Configuration manifold of a rigid body

Definition 3.6. *A **rigid body with a fixed point** O is a system of point masses where the condition $x_1 = O$ adds to conditions 3.2.*

Clearly, its configuration manifold is the three-dimensional rotation group $SO(3)$.

3.1.2 Conservation laws

Consider the problem of the motion of a free rigid body under its own inertia[2].

The system admits all translational displacements: they do not change the lagrangian function. Therefore by Noether's theorem there exist three first integrals, those are necessarily the three components of the vector of momentum. Therefore, we have shown

[1]Strictly speaking, the configuration space of a rigid body is $\mathbb{R}^3 \times O(3)$, and $\mathbb{R}^3 \times SO(3)$ is only one of the two connected components of this manifold, corresponding to the orientation of the body.

[2]That means, under the action of no force, or equivalently, without the presence of a potential function.

Theorem 3.7. *Under the free motion of a rigid body, its center of mass moves uniformly and linearly.*

Now we can look at an inertial coordinate system[3], in which the center of inertia is stationary. Then we have

Corollary 3.8. *A free rigid body rotates about its center of mass as if the center of mass were fixed at a stationary point O.*

In this way, the problem is reduced to the problem, with three degrees of freedom, of the motion of a rigid body around a fixed point O. We will study this problem in more detail (not necessarily assuming that O is the center of mass of the body).

The lagrangian function admits all rotations around O. By Noether's theorem there exist three corresponding first integrals: the three components of the vector of angular momentum. The total energy of the system, $E = T$, is also conserved (here it is equal to the kinetic energy). Therefore, we have shown

Theorem 3.9. *In the problem of the motion of a rigid body around a stationary point O, in the absence of outside forces, there are four first integrals: M_x, M_y, M_z, and E.*

From this theorem we can get qualitative conclusions about the motion without any calculation.

The position and velocity of the body are determined by a point in the six-dimensional manifold $TSO(3)$-the tangent bundle of the configuration manifold $SO(3)$. The first integrals M_x, M_y, M_z, and E are four functions on $TSO(3)$. One can verify that in the general case (if the body does not have any particular symmetry) these four functions are independent. Therefore, the four equations

$$M_x = C_1 \quad M_y = C_2 \quad M_z = C_3 \quad E = C_4 > 0$$

define a two dimensional submanifold V_c in the six-dimensional manifold $TSO(3)$.

This manifold is invariant: if the initial conditions of motion give a point on V_c, then for all time of the motion, the point in $TSO(3)$ corresponding to the position and velocity of the body remains in V_c.

Therefore, V_c admits a tangent vector field (namely, the field of velocities of the motion on $TSO(3)$); for $C_4 > 0$ this field cannot have singular points. Furthermore, it is easy to verify that V_c is compact (using E) and orientable (since $TSO(3)$ is orientable).

In topology it is proved that the only connected orientable compact two-dimensional manifolds are the spheres with n handles, $n \geq 0$ (Figure 3.2). Of these, only the torus ($n = 1$) admits a tangent vector field without singular points. Therefore, the invariant manifold V_c is a two-dimensional torus (or several tori).

We will see later that one can choose angular coordinates $\varphi_1, \varphi_2, \mod 2\pi$ on this torus such that a motion represented by a point of V_c is given by the equations $\dot{\varphi}_1 = \omega_1(c), \dot{\varphi}_2 = \omega_2(c)$. In other words, a rotation of a rigid body is represented by the superposition of two periodic motions with (usually) different periods: if the frequencies ω_1 and ω_2 are incommensurable, then the body never returns to its original state of motion. It does, nevertheless, return to an arbitrarily small neighborhood of its original state at some point. The magnitudes of the frequencies ω_1 and ω_2 depend on the initial conditions C.

[3]This may sound familiar from Einstein's theory of general relativity as well. We say that a coordinate system is inertial if moves without acceleration or rotation.

Figure 3.2: Two-dimensional compact connected orientable manifolds

3.1.3 The inertia operator

We now go on to the quantitative theory and introduce the following notation. Let k be a stationary coordinate system and K a coordinate system rotating together with the body around the point O: in K the body is at rest. Every vector in K is carried over to k by an operator B.

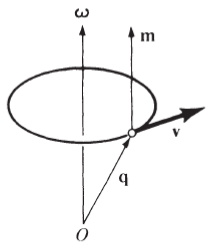

Figure 3.3: Radius vector and vectors of velocity, angular velocity and angular momentum of a point of the body in space

Corresponding vectors in K and k will be denoted by the same letter; capital for K and lower case for k. So, for example (Figure 3.3),

- $q \in k$ is the radius vector of a point in space;
- $Q \in K$ is its radius vector in the body, $q = BQ$;
- $v = \dot{q} \in k$ is the velocity vector of a point in space;
- $V \in K$ is the same vector in the body, $v = BV$;
- $\omega \in k$ is the angular velocity in space;
- $\Omega \in K$ is the angular velocity in the body, $\omega = B\Omega$;
- $m \in k$ is the angular momentum in space;
- $M \in K$ is the angular momentum in the body, $m = BM$.

Since the operator $B : K \to k$ preserves the metric and orientation, it preserves the scalar and vector products.

By definition of angular velocity,

$$v = [\omega, q].$$

By definition of the angular momentum of a point of mass m with respect to O,

$$m = [q, mv] = m[q, [\omega, q]].$$

Therefore,

$$M = m[Q, [\Omega, Q]].$$

Hence, there is a linear operator transforming Ω to M:

$$A : K \to K \quad A\Omega = M.$$

This operator still depends on a point of the body (Q) and its mass (m).

Lemma 3.10. *The operator A is symmetric.*

Proof. In view of the relation $([a, b], c) = ([c, a], b)$ we have, for any X and Y in K,

$$(AX, Y) = m([Q, [X, Q]], Y) = m([Y, Q], [X, Q]),$$

and the last expression is symmetric in X and Y. □

By substituting the vector of angular velocity Ω for X and Y and noticing that $[\Omega, Q]^2 = V^2 = v^2$, we obtain

Corollary 3.11. *The kinetic energy of a point of a body is a quadratic form with respect tot the vector of angular velocity Ω, namely:*

$$T = \frac{1}{2}(A\Omega, \Omega) = \frac{1}{2}(M, \Omega).$$

Definition 3.12. *The symmetric operator A is called the* ***inertia operator***[4] ***of the point*** *Q.*

If a body consists of many points Q_i with masses m_i, then by summing we obtain

Theorem 3.13. *The angular momentum M of a rigid body with respect to a stationary point O depends linearly on the angular velocity Ω, i.e., there exists a linear operator $A : K \to K$, $A\Omega = M$. The operator A is symmetric.*

The kinetic energy of a body is a quadratic form with respect to the angular velocity Ω,

$$T = \frac{1}{2}(A\Omega, \Omega) = \frac{1}{2}(M, \Omega).$$

Proof. By definition, the angular momentum of a body is equal to the sum of the angular momenta of its points:

$$M = \sum_i M_i = \sum_i A_i \Omega = A\Omega, \quad \text{where } A = \sum_i A_i.$$

Since by the lemma the inertia operator A_i of every point is symmetric, the operator A is also symmetric. For kinetic energy we obtain, by definition,

$$T = \sum_i T_i = \sum_i \frac{1}{2}(M_i, \Omega) = \frac{1}{2}(M, \Omega) = \frac{1}{2}(A\Omega, \Omega).$$

□

[4] Often called the inertia tensor.

3.1.4 Principal axes

Like every symmetric operator, A has three mutually orthogonal characteristic directions. Let e_1, e_2, and $e_3 \in K$ be their unit vector and I_1, I_2, and I_3 their eigenvalues. In the basis e_i, the inertia operator and the kinetic energy have a particularly simple form:

$$\begin{aligned} M_i &= I_i \Omega_i \\ T &= \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2). \end{aligned}$$

Definition 3.14. *The axes e_i are called the* ***principal axes*** *of the body at the point O.*

Finally, if the numbers I_1, I_2 and I_3 are not all different, then the axes e_i are not uniquely defined. We will further clarify the meaning of the eigenvalues I_1, I_2 and I_3.

Theorem 3.15. *For a rotation of a rigid body fixed at a point O, with angular velocity Ω around the e axis, the kinetic energy is equal to*

$$T = \frac{1}{2} I_e |\Omega|^2, \quad \textit{where } I_e = \sum_i m_i r_i^2$$

and r_i is the distance of the i-th point to the e axis (Figure 3.4).

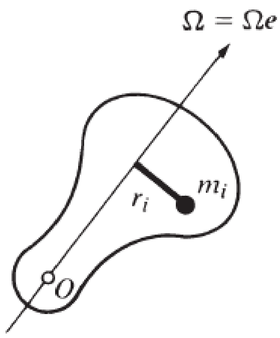

Figure 3.4: Kinetic energy of a body rotating around an axis

Proof. By definition $T = \frac{1}{2}\sum m_i v_i^2$; but $|v_i| = |\Omega| r_i$, so $T = \frac{1}{2}\left(\sum m_i r_i^2\right)|\Omega|^2$. □

The number I_e depends on the direction e of the axis of rotation ω in the body.

Definition 3.16. *I_e is called the* ***moment of inertia*** *of the body with respect to the e axis:*

$$I_e = \sum_i m_i r_i^2.$$

By comparing the two expressions for T we obtain:

Corollary 3.17. *The eigenvalues I_i of the inertia operator A are the moments of inertia of the body with respect to the principal axes e_i.*

3.1.5 The inertial ellipsoid

In order to study the dependence of the moment of inertia I_e upon the direction of the axis e in a body, we consider the vectors $e/\sqrt{I_e}$, where the unit vector e runs over the unit sphere

Theorem 3.18. *The vectors $e/\sqrt{I_e}$ form an ellipsoid in K.*

Proof. If $\Omega = e/\sqrt{I_e}$, then the quadratic form $T = \frac{1}{2}(A\Omega, \Omega)$ is equal to $\frac{1}{2}$. Therefore, $\{\Omega\}$ is the level set of a positive-definite quadratic form, i.e., an ellipsoid. $\square$

One could say that this ellipsoid consists of those angular velocity vectors Ω whose kinetic energy is equal to $\frac{1}{2}$.

Definition 3.19. *The ellipsoid $\{\Omega : (A\Omega, \Omega) = 1\}$ is called the* ***inertia ellipsoid of the body*** *at the point* 0 *(Figure 3.5).*

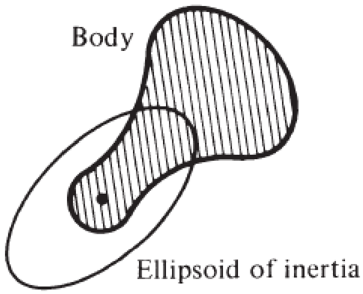

Figure 3.5: Ellipsoid of inertia

In terms of the principal axes e_i, the equation of the inertia ellipsoid has the form

$$I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = 1.$$

Therefore the principal axes of the inertia ellipsoid are directed along the principal axes of the inertia tensor, and their lengths are inversely proportional to $\sqrt{I_i}$.

Remark 3.20. If a body is stretched out along some axis, then the moment of inertia with respect to this axis is small, and consequently, the inertia ellipsoid is also stretched out along this axis; thus, the inertia ellipsoid may resemble the shape of the body.

If a body has an axis of symmetry of order k passing through O (so that it coincides with itself after rotation by $2\pi/k$ around the axis), then the inertia ellipsoid also has the same symmetry with respect to this axis. But a triaxial ellipsoid does not have axes of symmetry of order $k > 2$. Therefore, every axis of symmetry of a body of order $k > 2$ is an axis of rotation of the inertia ellipsoid and, therefore, a principal axis.

If there are several such axes, then the inertia ellipsoid is a sphere, and any axis is principal.

We now remark that the inertia ellipsoid (or the inertia operator or the moments of inertia I_1, I_2 and I_3) completely determines the rotational characteristics of our body: if we consider two bodies with identical inertia ellipsoids, then for identical initial conditions they will move identically (since they have the same lagrangian function $L = T$).

Therefore, from the point of view of the dynamics of rotations around 0, the space of all rigid bodies is three-dimensional, however many points compose the body.

We can even consider the "solid rigid body of density $\rho(Q)$." having in mind the limit as $\Delta Q \to 0$ of the sequence of bodies with a finite number of points Q_i with masses $\rho(Q_i)\Delta Q_i$ (Figure 3.6) or, what amounts to the same thing, any body with moments of inertia

$$I_e = \iiint \rho(Q) r^2(Q) dQ,$$

where r is the distance from Q to the e axis.

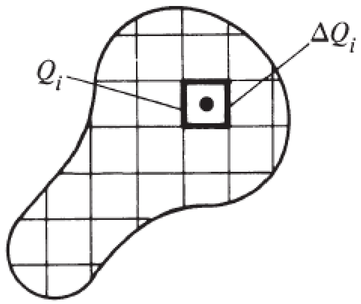

Figure 3.6: Continuous solid rigid body

3.2 Lagrange's top

3.2.1 Euler angles

Consider a rigid body fixed at a stationary point O and subject to the action of the gravitational force mg.

In this problem with three degrees of freedom, only two first integrals are known: the total energy $E = T + U$, and the third component of M on the cartesian coordinate system, M_z. There is an important special case in which the problem can be completely solved- the case of a symmetric top.

Definition 3.21. *A **symmetric** or **lagrangian top** is a rigid body fixed at a stationary point O whose inertia ellipsoid at O is an ellipsoid of revolution and whose center of gravity lies on the axis of symmetry* e_3 *(Figure 3.7).*

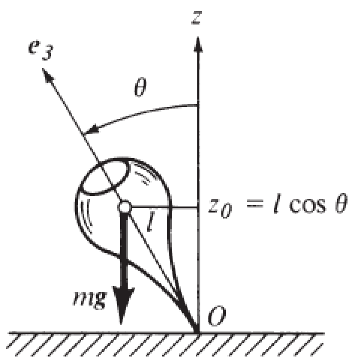

Figure 3.7: Lagrangian top

In this case, a rotation around the e_3 axis does not change the lagrangian function, and by Noether's theorem there must exist a first integral in addition to E and M_z (as we will see, it turns out to be the projection M_3 of the angular momentum vector on the e_3 axis).

If we can introduce three coordinates so that the angles of rotation around the z axis and around the axis of the top are among them, then these coordinates will be cyclic, and the problem with three degrees of freedom will reduce to a problem with one degree of freedom (for the third coordinate).

Such a choice of coordinates on the configuration space $SO(3)$ is possible; these coordinates φ, ψ, θ are called the **Euler angles** and they form a local coordinate system in $SO(3)$ similar to geographical coordinates on the sphere: they exclude the poles and are multiple-valued on one meridian. We introduce the following notation (Figure 3.8):

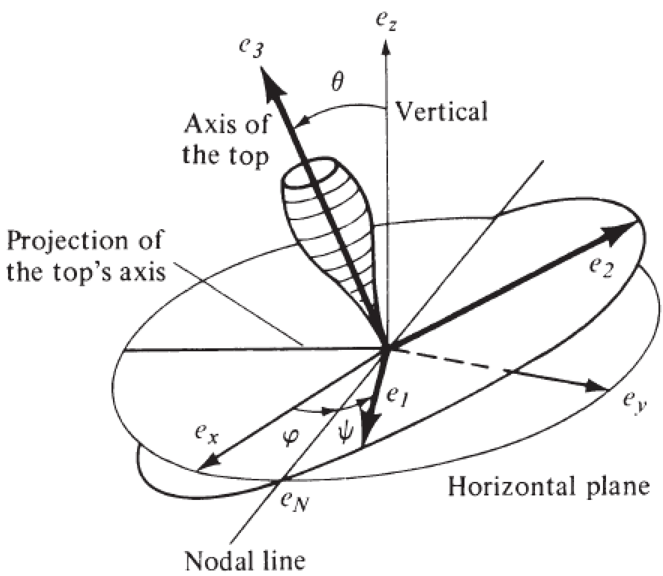

Figure 3.8: Euler angles

e_x, e_y and e_z	are the unit vectors of a right-handed cartesian stationary coordinate system at the stationary point O;
e_1, e_2 and e_3	are the unit vectors of a positive oriented coordinate system connected to the body, directed along the principal axes at O;
$I_1 = I_2 \neq I_3$	are the moments of inertia of the body at O;
e_N	is the unit vector of the axis $[e_z, e_3]$, called the **line of nodes** (all vectors are in the "stationary space" k).

In order to carry the stationary frame (e_x, e_y, e_z) into the moving frame (e_1, e_2, e_3), we must perform three rotations through

1. an angle φ around the e_z axis. Under this rotation, e_z remains fixed, and e_x goes to e_N.
2. an angle θ around the e_N axis. Under this rotation, e_z goes to e_3, and e_N remains fixed.
3. an angle ψ around the e_3 axis. Under this rotation, e_N goes to e_1, and e_3 stays fixed.

After all three rotations, e_x has gone to e_1, and e_z to e_3; therefore, e_y goes to e_2.

Definition 3.22. *The angles φ, ψ, and θ are called the **Euler angles**.*

Theorem 3.23. *To every triple of numbers φ, θ, ψ the construction above associates a rotation of three-dimensional space, $B(\varphi, \theta, \psi) \in SO(3)$, taking the frame (e_x, e_y, e_z) into the frame (e_1, e_2, e_3). In addition, the mapping $(\varphi, \theta, \psi) \to B(\varphi, \theta, \psi)$ gives local coordinates*

$$0 < \varphi < 2\pi, \quad 0 < \psi < 2\pi, \quad 0 < \theta < \pi$$

on $SO(3)$, the configuration space of the top. Like geographical longitude, φ and ψ can be considered as angles $\mod 2\pi$; for $\theta = 0$ or $\theta = \pi$ the map $(\varphi, \theta, \psi) \to B$ has a pole-type singularity.

3.2.2 Calculation of the lagrangian function

We will express the lagrangian function in terms of the coordinates φ, θ, ψ and $\dot{\varphi}, \dot{\theta}, \dot{\psi}$.

The potential energy is equal to

$$U = \iiint zg \, dm = mgz_0 = mgl \cos \theta,$$

where z_0 is the height of the center of gravity above 0.

We now calculate the kinetic energy. A small trick is useful here: we consider the particular case when $\varphi = \psi = 0$.

Lemma 3.24. *The angular velocity of a top is expressed in terms of the derivatives of the Euler angles by the formula*

$$\omega = \dot{\theta} e_1 + (\dot{\varphi} \sin \theta) e_2 + (\dot{\psi} + \dot{\varphi} \cos \theta) e_3,$$

if $\varphi = \psi = 0$.

Proof. We look at the velocity of a point of the top occupying the poisition r at time t. After time dt this point takes the position (within $(dt)^2$)

$$B(\varphi + d\varphi, \theta + d\theta, \psi + d\psi) B^{-1}(\varphi, \theta, \psi) r,$$

where $d\varphi = \dot{\varphi} \, dt, d\theta = \dot{\theta} \, dt$ and $d\psi = \dot{\psi} \, dt$.

Consequently, to the same accuracy the displacement vector is the sum of the three terms

$$B(\varphi + d\varphi, \theta, \psi) B^{-1}(\varphi, \theta, \psi) r - r = [\omega_\varphi, r] dt,$$
$$B(\varphi, \theta + d\theta, \psi) B^{-1}(\varphi, \theta, \psi) r - r = [\omega_\theta, r] dt,$$
$$B(\varphi, \theta, \psi + d\psi) B^{-1}(\varphi, \theta, \psi) r - r = [\omega_\psi, r] dt$$

(the angular velocities $\omega_\varphi, \omega_\theta$ and ω_ψ are defined by these formulas).

Therefore, the velocity of the point r is $v = [\omega_\varphi + \omega_\theta + \omega_\psi, r]$, so the angular velocity of the body is

$$\omega = \omega_\varphi + \omega_\theta + \omega_\psi,$$

where the terms are defined by the formulas above.

It remains to decompose the vectors $\omega_\varphi, \omega_\theta$, and ω_ψ with respect to e_1, e_2, and e_3. We have not yet used the fact that $\varphi = \psi = 0$. If $\varphi = \psi = 0$, then

$$B(\varphi + d\varphi, \theta, \psi) B^{-1}(\varphi, \theta, \psi)$$

is simply a rotation around the axis e_z through an angle $d\varphi$, so

$$\omega_\varphi = \dot{\varphi} e_z.$$

Furthermore, $B(\varphi, \theta + d\theta, \psi) B^{-1}(\varphi, \theta, \psi)$ is simply a rotation around the axis $e_N = e_x = e_1$ through an angle $d\theta$ in the case $\varphi = \psi = 0$, so

$$\omega_\theta = \dot{\theta} e_1.$$

Finally, $B(\varphi, \theta, \psi + d\psi) B^{-1}(\varphi, \theta, \psi)$ is a rotation through an angle $d\psi$ around the axis e_3, so

$$\omega_\psi = \dot{\psi} e_3.$$

In short, for $\varphi = \psi = 0$ we have

$$\omega = \dot{\varphi} e_z + \dot{\theta} e_1 + \dot{\psi} e_3.$$

But, clearly, for $\varphi = \psi = 0$

$$e_z = e_3 \cos\theta + e_2 \sin\theta.$$

So the components of the angular velocity along the principal axes e_1, e_2, and e_3 are

$$\omega_1 = \dot{\theta} \quad \omega_2 = \dot{\varphi} \sin\theta \quad \omega_3 = \dot{\psi} + \dot{\varphi} \cos\theta.$$

□

Since $T = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$, the kinetic energy for $\varphi = \psi = 0$ is given by the formula

$$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2.$$

But the kinetic energy cannot depend on φ and ψ: these are cyclic coordinates, and by choice of oirigin of reference for φ and ψ, which does not change T, we can always make $\varphi = 0$ and $\psi = 0$. Thus the formula we got for the kinetic energy is true for all φ and ψ.

In this way we obtain the lagrangian function

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2 - mgl\cos\theta.$$

3.2.3 Investigation of the motion

To the cyclic coordinates φ and ψ there correspond the first integrals

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\varphi}} &= M_z = \dot{\varphi}(I_1 \sin^2\theta + I_3 \cos^2\theta) + \dot{\psi} I_3 \cos\theta \\
\frac{\partial L}{\partial \dot{\psi}} &= M_3 = \dot{\varphi} I_3 \cos\theta + \dot{\psi} I_3.
\end{aligned}$$

Theorem 3.25. *The inclination θ of the axis of the top to the vertical changes with time in the same way as in the one-dimensional system with energy*

$$E' = \frac{I_1}{2}\dot{\theta}^2 + U_{eff}(\theta),$$

where the effective potential energy is given by the formula

$$U_{eff} = \frac{(M_z - M_3\cos\theta)^2}{2I_1 \sin^2\theta} + mgl\cos\theta.$$

Proof. Following the general theory, we express $\dot{\varphi}$ and $\dot{\psi}$ in terms of M_3 and M_z. We get the total energy of the system as

$$E = \frac{I_1}{2}\dot{\theta}^2 + \frac{M_3^2}{2I_3} + mgl\cos\theta + \frac{(M_z - M_3\cos\theta)^2}{2I_1\sin^2\theta}$$

and

$$\dot{\varphi} = \frac{M_z - M_3\cos\theta}{I_1\sin^2\theta}$$

The number $M_3^2/2I_3 = E - E'$, independent of θ, does not affect the equation for θ. $\square$

In order to study the one-dimensional system above it is convenient to make the substitution $\cos\theta = u\,(-1 \le u \le 1)$.

We also write

$$\frac{M_z}{I_1} = a, \quad \frac{M_3}{I_1} = b\frac{2E'}{I_1} = \alpha, \quad \frac{2mgl}{I_1} = \beta > 0.$$

Then we can rewrite the law of conservation of energy E' as

$$\dot{u}^2 = f(u),$$

where $f(u) = (\alpha - \beta u)(1-u^2) - (a - bu)^2$, and the law of variation of the azimuth φ as

$$\dot{\varphi} = \frac{a - bu}{1 - u^2}.$$

We notice that $f(u)$ is a polynomial of degree 3, $f(+\infty) = +\infty$, and $f(\pm 1) = -(a \mp b)^2 < 0$ if $a \neq \pm b$. On the other hand, actual motions correspond to constants a, b, α, and β for which $f(u) \ge 0$ for some $-1 \le u \le 1$. Thus $f(u)$ has exactly two real roots u_1 and u_2 on the interval $-1 \le u \le 1$ (and one for $u > 1$, Figure 3.9). Therefore, the inclination θ of the axis of the top changes periodically between two limit values θ_1 and θ_2.

Definition 3.26. *This periodic change in inclination is called* ***nutation****.*

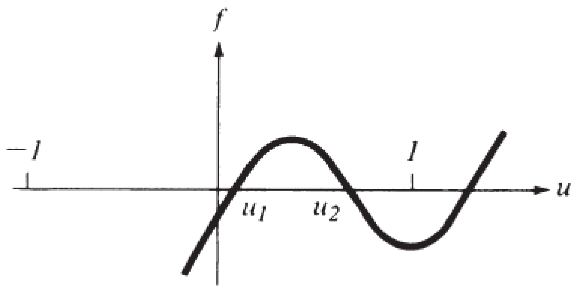

Figure 3.9: Graph of the function $f(u)$

We now consider the motion of the azimuth of the axis of the top. The point of intersection of the axis with the unit sphere moves in the ring between the parallels θ_1 and θ_2. The variation of the azimuth of the axis is determined by the equation

$$\dot{\varphi} = \frac{a - bu}{1 - u^2}.$$

If the root u' of the equation $a = bu$ lies outside of (u_1, u_2), then the angle φ varies monotonically and the axis traces a curve like a sinusoid on the unit sphere (Figure 3.10 (a)). If the root u' of the equation $a = bu$ lies inside (u_1, u_2), then the rate of change of φ is in opposite directions on the parallels θ_1 and θ_2, and the axis traces a looping curve in the sphere[5] (Figure 3.10 (b)).

[5] Similar to the orbits of the other planets around the Earth as thought by the Ptolemaic theory of epicicles.

If the root u' of $a = bu$ lies on the boundary (e.g., $u' = u_2$), then the axis traces a curve with cusps (Figure 3.10 (c)).

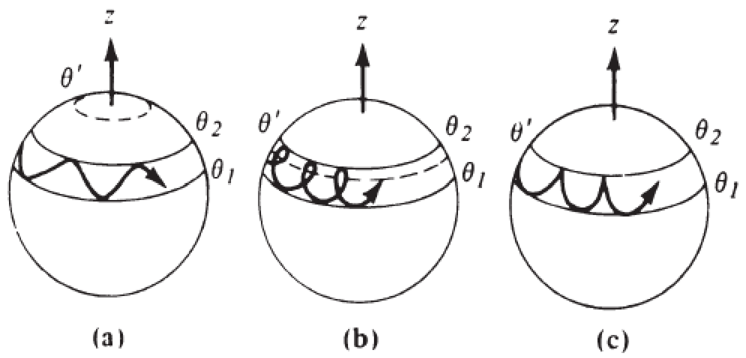

Figure 3.10: Path of the top's axis on the unit sphere

The last case, although exceptional, is observed every time we release the axis of a top launched at inclination θ_2 without initial velocity; the top first falls, but then rises again.

Definition 3.27. *The azimuthal motion of the top is called **precession**.*

The complete motion of the top consists of rotation around its own axis, nutation, and precession. Each of the three motions has its own frequency. If the frequencies are incommensurable, the top never returns to its initial position, although it approaches it arbitrarily closely.

Further reading

Section 31 of Arnold [2] gives some insights on the natural following direction, the particular cases named "sleeping" tops and "fast" tops.

Sections 5.8 and 5.9 of Goldstein [9] propose the application of the given formalism into interesting problems.

Chapter 4

Liouville-Arnold theorem and action-angle variables

In order to integrate a system of $2n$ ordinary differential equations, generally speaking we must know $2n$ first integrals. It turns out that if we are given a canonical system of differential equations, it is often sufficient to know only n first integrals -each of them allows us to reduce the order of the system not just by one, but by two.

4.1 Integrable systems

4.1.1 The Liouville-Arnold Theorem on integrable systems

Definition 4.1. We say that a function F is a **first integral** of a system with hamiltonian function H if and only if the Poisson bracket

$$\{H, F\} \equiv 0$$

is identically equal to zero.

Definition 4.2. Two functions F_1 and F_2 on a symplectic manifold are **in involution** if their Poisson bracket $\{F_1, F_2\} \equiv 0$ is equal to zero.

Liouville proved that if, in a system with n degrees of freedom (i.e., with a $2n$-dimensional phase space), n independent first integrals in involution are known, then the system is integrable by quadratures[1].

Theorem 4.3. *If we are given n functions in involution on a symplectic $2n$-dimensional manifold*

$$F_1 = H, F_2, \dots, F_n \quad \{F_i, F_j\} \equiv 0, \quad i, j = 1, 2, \dots, n.$$

Consider a level set of the functions F_i

$$M_f = \{x : F_i(x) = f_i,\ i = 1, \dots, n\}.$$

Assume that the n functions F_i are independent on M_f (i.e., the n 1-forms dF_i are linearly independent at each point of M_f). Then

[1] A system is said to be integrable by quadratures if it is possible to calculate its solution only through the computation of definite integrals and the inverse of functions (these belong to the set of so-called elementary operations).

1. *M_f is a smooth manifold, invariant under the phase flow with hamiltonian function $H = F_1$.*
2. *If the manifold M_f is compact and connected, then it is diffeomorphic to the n-dimensional torus*
$$T^n = \{(\varphi_1, \dots, \varphi_n) \mod 2\pi\}.$$
3. *The phase flow with hamiltonian function H determines a conditionally periodic motion on M_f, i.e., in angular coordinates $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$ we have*
$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{f}).$$
4. *The canonical equations with hamiltonian function H can be integrated by quadratures.*

Before proving this theorem, we note a few of its corollaries.

Corollary 4.4. *If, in a canonical system with two degrees of freedom, a first integral F is known which does not depend on the hamiltonian H, then the system is integrable by quadratures; a compact connected two-dimensional submanifold of the phase space $H = h$, $F = f$ is an invariant torus, and motion on it is conditionally periodic.*

Proof. F and H are in involution since F is a first integral of a system with hamiltonian function H. □

Example 4.5. As an example with three degrees of freedom, we consider a heavy symmetric Lagrange top fixed at a point on its axis. Three first integrals are immediately obvious: H, M_z, and M_3. It is easy to verify that the integrals M_z and M_3 are in involution. Furthermore, the manifold $H = h$ in the phase space is compact. Therefore, we can immediately say, without any calculations, that for the majority of initial conditions [2] the motion of the top is conditionally periodic: the phase trajectories fill up the three-dimensional torus $H = c_1$, $M_z = c_2$, $M_3 = c_3$. The corresponding three frequencies are called frequencies of fundamental rotation, precession, and nutation.

4.1.2 Beginning of the proof of the Liouville-Arnold theorem

We turn now to the proof of the theorem. Consider the level set of the integrals:
$$M_f = \{x : F_i = f_i, i = 1, \dots, n\}.$$
By hypothesis, the n 1-forms dF_i are linearly independent at each point of M_f; therefore, by the implicit function theorem, M_f is an n-dimensional submanifold of the $2n$-dimensional phase space.

Lemma 4.6. *On the n-dimensional manifold M_f there exist n tangent vector fields which commute with one another and which are linearly independent at every point.*

Proof. The symplectic structure of phase space defines an isomorphism taking 1-forms to vector fields. This isomorphism carries the 1-form dF_i to the field X_{F_i} of phase velocities of the system with hamiltonian function F_i. We will show that the n fields X_{F_i} are tangent to M_f, commute, and are independent.

[2] The singular level sets, where the integrals are not functionally independent, constitute the exception.

The independence of the X_{F_i} at every point of M_f follows from the independence of the dF_i and the nonsingularity of the isomorphism. The fields X_{F_i} commute with one another, since the Poisson brackets of their hamiltonian functions $\{F_i, F_j\}$ are identically 0. For the same reason, the derivative of the function F_i in the direction of the field X_{F_j} is equal to zero for any $i, j = 1, \dots, n$. Thus the fields X_{F_i} are tangent to M_f, and the Lemma is proved. $\square$

We notice that we have proved more than the Lemma, we have also proved:

1. The manifold M_f is invariant with respect to each of the n commuting phase flows g_i^t with hamiltonian functions $F_i : g_i^t g_j^s = g_j^s g_i^t$.
2. The manifold M_f is null (i.e., the 2-form ω is zero on $TM_f|_x$).

This is true since the n vectors $X_{F_i}|_x$ are skew-orthogonal to one another ($\{F_i, F_j\} \equiv 0$) and form a basis of the tangent plane to the manifold M_f at the point x.

4.1.3 Manifolds on which the action of the group $\mathbb{R}^n$ is transitive

We will now use the following topological proposition (the proof is completed in section 4.1.4).

Lemma 4.7. *Let M^n be a compact connected differentiable n-dimensional manifold, on which we are given n pairwise-commutative and linearly independent at each point vector fields. Then M^n is diffeomorphic to an n-dimensional torus.*

The proof begins like this:

We denote by $g_i^t, i = 1, \dots, n$, the one-parameter groups of diffeomorphisms of M corresponding to the n given vector fields. Since the fields commute, the groups g_i^t and g_j^s commute. Therefore, we can define an action g of commutative group $\mathbb{R}^n = \{t\}$ on the manifold M by setting

$$g^t : M \to M \quad g^t = g_1^{t_1} \dots g_n^{t_n}, \quad (t = (t_1, \dots, t_n) \in \mathbb{R}^n).$$

Clearly, $g^{t+s} = g^t g^s \in \mathbb{R}^n$. Now fix a point $x_0 \in M$. Then we have a map

$$g : \mathbb{R}^n \to M \quad g(t) = g^t x_0.$$

(The point x_0 moves along the trajectory of the first flow for time t_1, along the second flow for time t_2, etc.).

The map g of a sufficiently small neighborhood V of the point $0 \in \mathbb{R}^n$ gives a chart in a neighborhood of x_0: every point $x_0 \in M$ has a neighborhood $U(x_0 \in U \subset M)$ such that g maps V diffeomorphically onto U. In order to view that, apply the implicit function theorem and use the linear independence of the fields at x_0.

We note that the map $g : \mathbb{R}^n \to M^n$ cannot be one-to-one since M^n is compact and $\mathbb{R}^n$ is not. We will examine the set of pre-images of $x_0 \in M^n$.

Definition 4.8. *The **stationary group** of the point x_0 is the set Γ of points $t \in \mathbb{R}^n$ for which $g^t x_0 = x_0$.*

Corollary 4.9. Γ *is a subgroup of the group $\mathbb{R}^n$, independent of the point x_0.*

Proof. If $g^s x_0 = x_0$ and $g^t x_0 = x_0$, then $g^{s+t} x_0 = g^s g^t x_0 = g^s x_0 = x_0$ and $g^{-t} x_0 = g^{-t} g^t x_0 = x_0$. Therefore, Γ is a subgroup of $\mathbb{R}^n$. If $x = g^r x_0$ and $t \in \Gamma$, then $g^t x = g^{t+r} x_0 = g^r g^t x_0 = g^r x_0 = x$. $\square$

In this way the stationary group Γ is a well-defined subgroup of $\mathbb{R}^n$ independent of the point x_0. In particular, the point $t = 0$ clearly belongs to Γ.

Using the fact that the map $g : V \to U$ is a diffeomorphism, one can see that, in a sufficiently small neighborhood V of the point $0 \in \mathbb{R}^n$, there is no point of the stationary group other than $t = 0$.

It can be also be seen that, in a neighborhood $t + V$ of any point $t \in \Gamma \subset \mathbb{R}^n$, there is no point of the stationary group Γ other than $t = 0$.

The points of the stationary group Γ lie in $\mathbb{R}^n$ discretely. Such subgroups are called **discrete subgroups**.

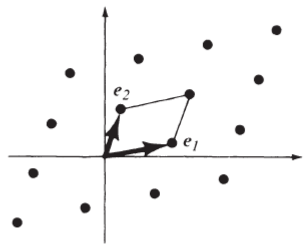

Figure 4.1: A discrete subgroup of the plane

Example 4.10. Let $e_1, \ldots, e_k$ be k linearly independent vectors in $\mathbb{R}^n$, $0 \le k \le n$. The set of all their integral linear combinations (Figure 4.1)

$$m_1 e_1 + \ldots + m_k e_k, \quad m_i \in \mathbb{Z}$$

forms a discrete subgroups of $\mathbb{R}^n$. For example, the set of all integral points in the plane is a discrete subgroup of the plane.

4.1.4 Discrete subgroups in $\mathbb{R}^n$

We will now use the algebraic fact that the example above includes all discrete subgroups of $\mathbb{R}^n$. More precisely, we will prove

Lemma 4.11. *Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exist k ($0 \le k \le n$) linearly independent vectors $e_1, \ldots, e_k \in \Gamma$ such that Γ is exactly the set of all their integral linear combinations.*

Proof. We will consider $\mathbb{R}^n$ with some euclidean structure. We always have $0 \in \Gamma$. If $\Gamma = \{0\}$ the lemma is proved. If not, there is a point $e_0 \in \Gamma$, $e_0 \neq 0$ (Figure 4.2). Consider the line $\mathbb{R}e_0$. We will show that among the elements of Γ on this line, there is a point e_1 which is closest to 0. In fact, in the disk of radius $|e_0|$ with center 0, there are only a finite number of points of Γ (as we saw above, every point x of Γ has a neighborhood V of standard size which does not contain any other point of Γ). Among the finite number of points of Γ inside this disc and lying on the line $\mathbb{R}e_0$, the point closest to 0 will be the closest point to 0 on the whole line. The integral multiples of this point e_1 ($me_1, m \in \mathbb{Z}$) constitute the intersection of the line $\mathbb{R}e_0$ with Γ. In fact, the points me_1 divide the line into pieces of length $|e_1|$. If there were a point $e \in \Gamma$ inside one of

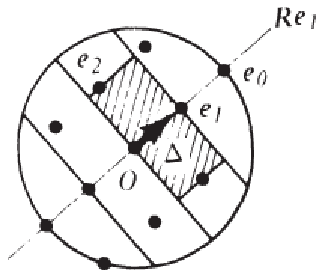

Figure 4.2: Proof of the lemma on discrete subgroups

these pieces $(me_1, (m+1)e_1)$, then the point $e - me_1 \in \Gamma$ would be closer to 0 than e_1. If there are no points of Γ off the line $\mathbb{R}e_1$, the lemma is proved. Suppose there is a point $e \in \Gamma, e \notin \mathbb{R}e_1$. We will show that there is a point $e_2 \in \Gamma$ closest to the line $\mathbb{R}e_1$ (but not lying on the line). We project e orthogonally onto $\mathbb{R}e_1$. The projection lies in exactly one interval $\Delta = \{\lambda e_1\}$, $m \leq \lambda \leq m+1$. Consider the right circular cylinder C with axis Δ and radius equal to the distance from Δ to e. In this cylinder lie a finite (nonempty) number of points of the group Γ. Let e_2 be the closest one to the axis $\mathbb{R}e_1$ not lying on the axis.

Using a shift of me_1, the projection of e could be moved onto the axis interval Δ. Then it could be seen that the distance from the axis to any point e of Γ not lying on $\mathbb{R}e_1$ is greater than or equal to the distance of e_2 from $\mathbb{R}e_1$.

The integral linear combinations of e_1 and e_2 form a lattice in the plane $\mathbb{R}e_1 + \mathbb{R}e_2$.

We claim that there are no points of Γ on the plane $\mathbb{R}e_1 + \mathbb{R}e_2$ other than integral linear combinations of e_1 and e_2. We can see that with a partition of the plane into parallelograms (Figure 4.3) $\Delta = \{\lambda_1 e_1 + \lambda_2 e_2\}$, $m_i \leq \lambda_i \leq m_i + 1$. If there were an $e \in \Delta$ with $e \neq m_1 e_1 + m_2 e_2$, then the point $e - m_1 e_1 - m_2 e_2$ would be closer to $\mathbb{R}e_1$ than e_2.

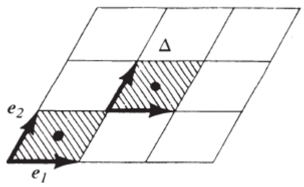

Figure 4.3: Partition of the plane into parallelograms

If there are no points of Γ outside the plane $\mathbb{R}e_1 + \mathbb{R}e_2$, the lemma is proved. Suppose that there is a point $e \in \Gamma$ outside this plane. Then there exists a point $e_3 \in \Gamma$ closest to $\mathbb{R}e_1 + \mathbb{R}e_2$; the points $m_1 e_1 + m_2 e_2 + m_3 e_3$ exhaust Γ in the three-dimensional space $\mathbb{R}e_1 + \mathbb{R}e_2 + \mathbb{R}e_3$. If Γ is not exhausted by these, we take the closest point to this three-dimensional space, etc. It can be seen that this closest point always exists.

Note that the vectors $e_1, e_2, e_3, \ldots$ are linearly independent. Since they all lie in $\mathbb{R}^n$, there are $k \leq n$ of them.

Partitioning the plane $\mathbb{R}e_1 + \ldots + \mathbb{R}e_k$ into parallelepipeds Δ and showing that there cannot be a point of Γ in any Δ one could show that Γ is exhausted by the integral linear combinations of $e_1, \ldots, e_k$. It should be taken into account that if there was an $e \in \Gamma$ outside the plane $\mathbb{R}e_1 + \ldots + \mathbb{R}e_k$, the construction would not be finished.

And thus Lemma 4.11 is proved. □

It is easy to prove Lemma 4.7: M_f is diffeomorphic to a torus T^n.

Consider the direct product of k circles and $n-k$ straight lines:

$$T^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \ldots, \varphi_k; y_1, \ldots, y_{n-k})\}, \quad \varphi \mod 2\pi,$$

together with the natural map $p : \mathbb{R}^{2n} \to T^k \times \mathbb{R}^{n-k}$,

$$p(\varphi, y) = (\varphi \mod 2\pi, y).$$

The points $f_1, \ldots, f_k \in \mathbb{R}^n$ (f_i has coordinates $\varphi_i = 2\pi, \varphi_j = 0, y = 0$) are mapped to 0 under this map.

Let $e_1, \ldots, e_k \in \Gamma \subset \mathbb{R}^n$ be the generators of the group Γ (cf. Lemma 4.11). We map the vector space $\mathbb{R}^n = \{(\varphi, y)\}$ onto the space $\mathbb{R}^n = \{t\}$ so that the vectors f_i go to e_i. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be such an isomorphism.

We now note that $\mathbb{R}^n = \{(\varphi, y)\}$ gives charts for $T^k \times \mathbb{R}^{n-k}$, and $\mathbb{R}^n = \{t\}$ gives charts for our manifold M_f. One could see that the map of charts $A : \mathbb{R}^n \to \mathbb{R}^n$ gives a diffeomorphism $\tilde{A} : T^k \times \mathbb{R}^{n-k} \to M_f$.

But, since the manifold M_f is compact by hypothesis, $k = n$ and M_f is an n-dimensional torus. Lemma 4.7 is proved.

In view of Lemma 4.6, the first two statements of the theorem are proved. At the same time, we have constructed angular coordinates $\varphi_1, \ldots, \varphi_n \mod 2\pi$ on M_f.

Under the action of the phase flow with hamiltonian H the angular coordinates φ vary uniformly with time

$$\dot{\varphi}_i = \omega_i \quad \omega_i = \omega_i(f) \quad \varphi(t) = \varphi(0) + \omega t.$$

In other words, motion in the invariant torus M_f is conditionally periodic.

Of all the assertions of the theorem, only the last remains to be proved: that the system can be integrated by quadratures.

4.2 Action-angle variables

4.2.1 Description of action-angle variables

In section 4.1 we studied one particular compact connected level manifold of the integrals: $M_f = \{x : F(x) = f\}$; it turned out that M_f was an n-dimensional torus, invariant with respect to the phase flow. We chose angular coordinates φ_i on M so that the phase flow with hamiltonian function $H = F_q$ takes an especially simple form:

$$\frac{d\varphi}{dt} = \omega(f) \quad \varphi(t) = \varphi(0) + \omega t.$$

We will now look at a neighborhood of the n-dimensional manifold M_f in $2n$-dimensional phase space.

Indeed, the manifold M_f has a neighborhood diffeomorphic to the direct product of the n-dimensional torus T^n and the disc D^n in n-dimensional euclidean space.

In the coordinates (F, φ) the phase flow with hamiltonian function $H = F_1$ can be written in the form of the simple system of $2n$ ordinary differential equations

$$\frac{dF}{dt} = 0 \quad \frac{d\varphi}{dt} = \omega(F), \tag{4.1}$$

which is easily integrated: $F(t) = F(0), \varphi(t) = \varphi(0) + \omega(F(0))t$.

Thus, in order to integrate explicitly the original canonical system of differential equations, it is sufficient to find the variables φ in explicit form. It turns out that this can be done using only quadratures.

A construction of the variables φ is given below.

We note that the variables (F, φ) are not, in general, symplectic coordinates. It turns out that there are functions of F, which we will denote by $I = I(F), I = (I_1, \ldots, I_n)$, such that the variables (I, φ) are symplectic coordinates: the original symplectic structure ω is expressed in them by the usual formula

$$\omega = \sum dI_i \wedge d\varphi_i.$$

The variables I are called action variables[3]; together with the angle variables φ they form the **action-angle system of canonical coordinates** in a neighborhood of M_f.

The quantities I_i are first integrals of the system with hamiltonian function $H = F_1$, since they are functions of the first integrals F_j. In turn, the variables F_i can be expressed in terms of I and, in particular, $H = F_1 = H(I)$. In action-angle variables the differential equations of our flow 4.1 have the form

$$\frac{dI}{dt} = 0 \quad \frac{d\varphi}{dt} = \omega(I). \tag{4.2}$$

Remark 4.12. In the variables (I, φ), the equations of the flow (4.2) have the canonical form with hamiltonian function $H(I)$. Therefore, $\omega(I) = \partial H/\partial I$; thus if the number of degrees of freedom is $n \geq 2$, the functions $\omega(I)$ are not arbitrary, but satisfy the symmetry condition $\partial\omega_i/\partial I_j = \partial\omega_j/\partial I_i$.

4.2.2 Construction of action-angle variables in the case of one degree of freedom

A system with one degree of freedom in the phase plane (p, q) is given by the hamiltonian function $H(p, q)$.

Example 4.13. The harmonic oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$; or, more generally, $H = \frac{1}{2}a^2p^2 + \frac{1}{2}b^2q^2$.

Example 4.14. The mathematical pendulum $H = \frac{1}{2}p^2 - \cos q$. In both cases we have a compact closed curve $M_h(H = h)$, and the conditions of the theorem of Section 4.1 for $n = 1$ are satisfied.

In order to construct the action-angle variables, we will look for a canonical transformation $(p, q) \to (I, \varphi)$ satisfying the two conditions:

$$I = I(h) \tag{4.3}$$

$$\oint_{M_h} d\varphi = 2\pi.$$

Example 4.15. In the case of the simple harmonic oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$, if r, φ are polar coordinates, then $dp \wedge dq = r\,dr \wedge d\varphi = d(r^2/2) \wedge d\varphi$. Therefore, $I = H = (p^2 + q^2)/2$.

In order to construct the canonical transformation $p, q \to I, \varphi$ in the general case, we will look for its generating function $S(I, q)$:

$$p = \frac{\partial S(I, q)}{\partial q} \quad \varphi = \frac{\partial S(I, q)}{\partial I} \quad H\left(\frac{\partial S(I, q)}{\partial q}, q\right) = h(I). \tag{4.4}$$

[3] It is not hard to see that I has the dimensions of action.

We first assume that the function $h(I)$ is known and invertible, so that every curve M_h is determined by the value of I ($M_h = M_{h(I)}$). Then for a fixed value of I we have from (4.4)

$$dS|_{I=const} = p\,dq.$$

This relation determines a well-defined differential 1-form dS on the curve $M_{h(I)}$.

Integrating this 1-form on the curve $M_{h(I)}$ we obtain (in a neighborhood of a point q_0) a function

$$S(I,q) = \int_{q_0}^{q} p\,dq.$$

This function will be the generating function of the transformation (4.4) in a neighborhood of the point (I, q_0). The first of the conditions (4.3) is satisfied automatically: $I = I(h)$. To verify the second condition, we consider the behavior of $S(I,q)$ "in the large". After a circuit of the closed curve $M_{h(I)}$ the integral of $p\,dq$ increases by

$$\Delta S(I) = \oint_{M_{h(I)}} p\,dq,$$

equal to the area Π enclosed by the curve $M_{h(I)}$. Therefore, the function S is a "multiple-valued function" on $M_{h(I)}$: it is determined up to addition of integral multiples of Π. This term has no effect on the derivative $\partial S(I,q)/\partial q$; but it leads to the multi-valuedness of $\varphi = \partial S/\partial I$. This derivative turns out to be defined only up to multiples of $d\Delta S(I)/dI$. More precisely, the formulas (4.4) define a 1-form $d\varphi$ on the curve $M_{h(I)}$, and the integral of this form on $M_{h(I)}$ is equal to $d\Delta S(I)/dI$.

In order to fulfill the second condition, $\oint_{M_h} d\varphi = 2\pi$, we need that

$$\frac{d}{dI}\Delta S(I) = 2\pi, \quad I = \frac{\Delta S}{2\pi} = \frac{\Pi}{2\pi},$$

where $\Pi = \oint_{M_h} p\,dq$ is the area bounded by the phase curve $H = h$.

Definition 4.16. *The* ***action variable*** *in the one-dimensional problem with hamiltonian function $H(p,q)$ is the quantity $I(h) = (1/2\pi)\Pi(h)$.*

Finally, we arrive at the following conclusion. Let $d\Pi/dh \neq 0$. Then the inverse $I(h)$ of the function $h(I)$ is defined.

Theorem 4.17. *Set $S(I,q) = \int_{q_0}^{q} p\,dq|_{H=h(I)}$. Then formulas (4.4) give a canonical transformation $p, q \to I, \varphi$ satisfying conditions (4.3).*

Thus, the action-angle variables in the one-dimensional case are constructed.

Example 4.18. Let's find S and I for the harmonic oscillator: If $H = \frac{1}{2}a^2p^2 + \frac{1}{2}b^2q^2$, then M_h is the ellipse bounding the area $\Pi(h) = \pi(\sqrt{2h}/a)(\sqrt{2h}/b) = 2\pi h/\omega$. Thus for a harmonic oscillator the action variable is the ratio of energy to frequency. The angle variable φ is, of course, the phase of oscillation. Furthermore, in action-angle variables the equations of motion (4.2) give

$$\dot{\varphi} = \frac{\partial H}{\partial I} = \left(\frac{dI}{dh}\right)^{-1} = 2\pi\left(\frac{d\Pi}{dh}\right)^{-1} \quad T = \frac{2\pi}{\dot{\varphi}} = \frac{d\Pi}{dh}.$$

4.2.3 Construction of action-angle variables in $\mathbb{R}^{2n}$

We turn now to systems with n degrees of freedom given in $\mathbb{R}^{2n} = \{(p,q)\}$ by a hamiltonian function $H(p,q)$ and having n first integrals in involution $F_1 = H, F_2, \ldots, F_n$. We will not repeat the reasoning which brought us to the choice of $2\pi I = \oint p\,dq$ in the one-dimensional case, but will immediately define n action variables I.

Let $\gamma_1, \ldots, \gamma_n$ be a basis for the one-dimensional cycles on the torus M_f (the increase of the coordinate φ_i on the cycle γ_j is equal to 2π if $i = j$ and 0 if $i \neq j$). We set

$$I_i(f) = \frac{1}{2\pi} \oint_{\gamma_i} p\,dq. \tag{4.5}$$

Using Stoke's formula one could check that this integral does not depend on the coice of the curve γ_i representing the cycle.

Definition 4.19. *The n quantities $I_i(f)$ given by formula (4.5) are called* ***action variables****.*

We assume now that, for the given values f_i on the n integrals F_i, the n quantities I_i are independent: $\det(\partial I/\partial f)|_f \neq 0$. Then in a neighborhood of the torus M_f we can take the variables I, φ as coordinates.

Theorem 4.20. *The transformation $p, q \to I, \varphi$ is canonical, i.e.*

$$\sum dp_i \wedge dq_i = \sum dI_i \wedge d\varphi_i.$$

Proof. We outline the proof of this theorem. Consider the differential 1-form $p\,dq$ on M_f. Since the manifold M_f is null (Section 4.1) this 1-form on M_f is closed: its exterior derivative $\omega = dp \wedge dq$ is identically equal to zero on M_f. Therefore,

$$S(x) = \int_{x_0}^{x} p\,dq|_{M_f}$$

does not change under deformations of the path of integration (Stoke's formula). Thus $S(x)$ is a "multiple-valued function" on M_f, with periods equal to

$$\Delta_i S = \int_{\gamma_i} dS = 2\pi I_i.$$

Now, let x_0 be a point on M_f, in a neighborhood of which the n variables q are coordinates on M_f, such that the submanifold $M_f \subset \mathbb{R}^{2n}$ is given by n equations of the form $p = p(I,q), q(x_0) = q_0$. In a simply connected neighborhood of the point q_0 a single-valued function is defined,

$$S(I,q) = \int_{q_0}^{q} p(I,q)dq,$$

and we can use it as the generating function of a canonical transformation $p, q \to I, \varphi$:

$$p = \frac{\partial S}{\partial q} \quad \varphi = \frac{\partial S}{\partial I}.$$

It is not difficult to verify that these formulas actually give a canonical transformation, not only in a neighborhood of the point under consideration, but also "in the large" in a neighborhood of M_f. The coordinates φ will be multiple-valued with periods

$$\Delta_i \varphi_j = \Delta_i \frac{\partial S}{\partial I_j} = \frac{\partial}{\partial I_j} \Delta_i S = \frac{\partial}{\partial I_j} 2\pi I_i = 2\pi \delta_{ij},$$

as was to be shown. □

We note that all our constructions involve only "algebraic" operations (inverting functions) and "quadrature" -calculation of the integrals of known functions. In this way the problem of integrating a canonical system with $2n$ equations, of which n first integrals in involution are known, is solved by quadratures.

Now we can finally say that we have proved **Theorem 4.3**, the Liouville-Arnold theorem on completely integrable systems.

Remark 4.21. Even in the one-dimensional case the action-angle variables are not uniquely defined by the conditions (4.3). We could have taken $I' = I + const$ for the action variable and $\varphi' = \varphi + c(I)$ for the angle variable.

Remark 4.22. We constructed action-angle variables for systems with phase space $\mathbb{R}^{2n}$. We could also have introduced action-angle variables for a system on an arbitrary symplectic manifold. We restrict ourselves here to one simple example (Figure 4.4)

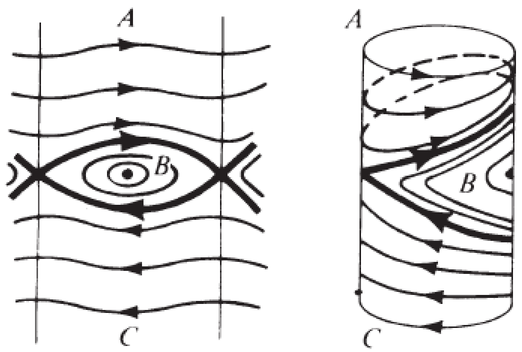

Figure 4.4: Action-angle variables on a symplectic manifold

Example 4.23. We could have taken the phase space of a pendulum ($H = \frac{1}{2}p^2 - \cos q$) to be, instead of the plane $\{(p,q)\}$, the surface of the cylinder $\mathbb{R}^1 \times S^1$ obtained by identifying angles q differing by an integral multiple of 2π.

The critical lines $H = \pm 1$ divide the cylinder into three parts, A, B and C, each of which is diffeomorphic to the direct product $\mathbb{R}^1 \times S^1$. We can introduce action-angle variables into each part. In the bounded part (B) the closed trajectories represent the oscillation of the pendulum; in the unbounded parts they represent rotation.

Remark 4.24. In the general case, as in the example analyzed above, the equations $F_i = f_i$ cease to be independent for some values of f_i, and M_f ceases to be a manifold. Such critical values of f correspond to separatrices dividing the phase space of the integrable problem into parts corresponding to the parts A, B, and C above. In some of these parts the manifolds M_f can be unbounded (parts A and C in the plane $\{(p,q)\}$); others are stratified into n-dimensional invariant tori M_f; in a neighborhood of such a torus we can introduce action-angle variables.

Further reading

An alternative statement and proof of the Liouville-Arnold theorem can be found in section 1.4 of Bolsinov and Fomenko [5].

Accessible examples with explicit calculation of action-angle variables will be found in the article [4], together with a truly nice introduction to the current state of research, and some further steps in the direction of perturbation theory.

Chapter 5

The Kepler problem

The Kepler problem is a special case of the two-body problem, in which two bodies interact by a central force which depends on the inverse squared of the distance between them. Generally the system is studied from the reference system of the center of mass. Some considerations are usually made in an implicit way: usually one would understand one of the two bodies being way more massive than the other one. Under this assumption, the reference system of the center of mass closely resembles a reference system where the more massive body remains at rest.

5.1 First Integrals of the system

5.1.1 Central force: conservation of angular momentum

Suppose we are given a Hamiltonian dynamical system H on $T^*\mathbb{R}^3$. Define the **angular momentum** by

$$L := q \times p.$$

Consider the Hamiltonian

$$H = \frac{1}{2}|p|^2 + V(\|q\|) \tag{5.1}$$

on $\mathbb{R}^{2n} - \{0\} \times \mathbb{R}^{2n}$, where $V : \mathbb{R} \to \mathbb{R}$ is a (smooth) function, possibly with some singularities. Such function V is called the potential for a **central force**, because it only depends on the distance.

We will assume that $n = 3$, although this can be generalized.

Lemma 5.1. *The angular momentum is preserved under the flow of X_H. In other words, the components of the angular momentum $L = (L_1, L_2, L_3)$ satisfy $\{H, L_i\} = 0$.*

Proof. The Hamiltonian for a central force is $SO(n)$-invariant, so Noether's theorem, implies the claim. □

Remark 5.2. The physical interpretation of preservation of angular momentum is that flow lines of the Hamiltonian vector field X_H lie in the plane with normal vector L.

We shall consider the Hamiltonian

$$H = \frac{1}{2}|p|^2 - \frac{1}{|q|} \tag{5.2}$$

on $\mathbb{R}^n - \{0\} \times \mathbb{R}^n$ with coordinates (q, p) and symplectic form $\omega = dp \wedge dq$. The physically relevant cases are $n = 2, 3$, and we shall first consider the case $n = 3$. The equations of motion are

$$\begin{aligned} \dot{p} &= -\frac{q}{|q|^3} \\ \dot{q} &= p. \end{aligned}$$

In other words, the force equals $\ddot{q} = -\frac{q}{|q|^3}$, so its strength drops off with the distance squared.

The strategy is to find as many integrals as possible, and in fact the Kepler problem turns *completely integrable*.

Lemma 5.3. *The angular momentum L is an integral of the Kepler problem.*

The Kepler problem has an obvious $SO(3)$-symmetry, or put it differently, the force is central, so Lemma 5.1 applies.

Remark 5.4. In higher dimensions this Hamiltonian has a $SO(n)$-symmetry, but we will not consider this more general (and unphysical) situation.

5.1.2 The Runge-Lenz vector

The following integral depends on the specific form of Kepler's hamiltonian (5.2).

Definition 5.5. *Define the **Laplace-Runge-Lenz vector** (also called **Runge-Lenz vector**) by*

$$A := p \times L - \frac{q}{|q|}.$$

Lemma 5.6. *The Runge-Lenz vector A is preserved under the flow of X_H. In other words, the components of $A = (A_1, A_2, A_3)$ satisfy $\{H, A_i\} = 0$.*

Unlike the preservation of angular momentum, this integral is not obvious from a symmetry of the phase space. We will prove that A is an integral with a short computation.

Proof. We compute the time-derivative of A,

$$\begin{aligned} \dot{A} &= [\dot{p}, L] + [p, \dot{L}] - \frac{\dot{q}}{\|q\|} + \frac{q}{\|q\|^2} \frac{q \cdot \dot{q}}{\|q\|} \\ &= -\left[\frac{q}{\|q\|^3}, [q, p] \right] - \frac{p}{\|q\|} + \frac{q}{\|q\|^3}(q \cdot p) \\ &= \frac{1}{\|q\|^3} \left[-q, [q, p] \right] - (q \cdot q)p + (q \cdot p)p) = 0. \end{aligned}$$

In the second step we have used the Hamilton equations, and in the last step we used the vector product identity

$$[[u, v], w] = (u \cdot w)v = -(v \cdot w)u.$$

$\square$

Lemma 5.7. *The Runge-Lenz vector satisfies the identity*

$$\|A\|^2 = 1 + 2H \cdot q \, \|L\|^2 . \tag{5.3}$$

Proof. The following computation makes use of the fact that p and L are orthogonal and the identity $q \cdot [p, L] = \det(q, p, L) = [q, p] \cdot L$. We find

$$\begin{aligned} \|A\|^2 &= \|[p,L]\|^2 - \frac{2}{|q|} q \cdot [p,L] + \frac{\|q\|^2}{\|q\|^2} = 1 + \|p\|^2 \|L\|^2 - \frac{2}{|q|} \|L\|^2 \\ &= 1 + 2\left(\frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}\right)\|L\|^2 . \end{aligned}$$

□

5.1.3 Solving the Kepler problem

Define the plane $P_L = \{v \in \mathbb{R}^3 | \langle L, v\rangle = 0\}$.

Lemma 5.8. *The vector A lies in the plane P_L. Recall that $\langle q, L\rangle = 0$ and observe that*

$$\langle A, L\rangle = \langle [p, L], L\rangle - \langle \frac{q}{\|q\|}, L\rangle = 0 + 0.$$

To describe the movement of the particle more explicitly, we apply a coordinates change, namely a rotation to move the L-vector to the z-axis. Then $L = (0, 0, l)$ for some $l > 0$, and hence we can write

$$A = (\|A\| \cos g, \|A\| \sin g, 0).$$

Definition 5.9. *The angle g is called the **argument of the perigee (perihelion)*** [1]

We now determine the radius as function of the angle ϕ. Using the above formula for A and the identity $\langle [p, L], q\rangle = \det(p, L, q)$, we find

$$\|q\| + \langle A, q\rangle = \langle \frac{q}{|q|}\rangle + \langle A, q\rangle = \langle [q, L], q\rangle = \det(p, L, q) = \langle [q, p], L\rangle = \|L\|^2 .$$

We write q in polar coordinates

$$q = (r \cos\phi, r \sin\phi, 0)$$

and by plugging this into $\|q\| + \langle A, q\rangle = \|L\|^2$, we find

$$r = \frac{\|L\|^2}{1 + \|A\| \cos(\phi - g)}. \tag{5.4}$$

It is common to call the quantity

$$f := \phi - g$$

the **true anomaly**, and $\|A\|$ is called the **eccentricity**. It is clear from equation (5.4) that the argument of the perigee is the angle of the closest approach in a typical situation (here this means $L \neq 0$).

With the above computations, we can deduce the following classification result for solutions.

[1] Perigee means close to the Earth. Perihelion means close to the Sun. If the heavy mass describes the Earth, one uses perigee, if it is the Sun, one uses the word perihelion.

Theorem 5.10. *Solutions to the Kepler problem are conic sections, i.e. curves that are the intersection of a plane and a cone.*

Proof. We intersect the plane given by $z = \|l\|^2 - \|K\| \cdot x$ with the cone given by $z = \sqrt{x^2 + y^2}$. This gives the equation

$$\sqrt{x^2 + y^2} = \|L\|^2 - \|K\| \cdot x,$$

which is equivalent to the above equation for a solution of the Kepler problem,

$$\|q\| + \langle K, q \rangle = \|L\|^2$$

if we substitute $\sqrt{x^2 + y^2} = \|q\|$ and $\langle K, q \rangle = \|K\| \, r \cos\phi = \|K\| \cdot x$. $\square$

5.2 The Planar Kepler problem

We have so far given a discussion on the spatial Kepler problem, the planar one follows as a special case. Later on we are mostly interested in the planar restricted three body problem. Therefore it will be helpful to develop some explicit formulas for the planar one.

The Hamiltonian for the planar Kepler problem is given by

$$E : T^*(\mathbb{R}^2 \backslash \{0\}) \to \mathbb{R}, \quad (q, p) \mapsto \frac{1}{2} p^2 - \frac{1}{|q|}. \tag{5.5}$$

We prefer to abbreviate the Hamiltonian for the Kepler problem by E, as an abbreviation for energy and not by H as usual. This is because we also have to study the Kepler problem in rotating coordinates, the so called rotating Kepler problem, and we prefer to save the letter H to denote the Hamiltonian in rotating coordinates.

In the spatial Kepler problem angular momentum is a three dimensional vector. In the planar case the first two components of this vector vanish and only the third component survives. If we talk about angular momentum for the planar Kepler problem we just mean this third component, hence

$$L : T^*\mathbb{R}^2 \to \mathbb{R}, \quad (q, p) \mapsto q_1 p_2 - q_2 p_1.$$

The Kepler problem is rotationally invariant. Because rotation is generated by angular momentum we obtain by Noether's theorem

$$\{E, L\} = 0, \tag{5.6}$$

as we discussed already in Lemma 5.3. Because the phase space of the planar Kepler problem is four dimensional it follows that E together with L are a completely integrable system. It is not hard to see in this example the invariant tori as predicted in the Liouville-Arnold theorem, namely **Theorem 4.3**. Let us discuss this for negative energy. In this case orbits of the Kepler problem are either ellipses or collision orbits. An ellipse is topologically a circle and if we rotate it we get a torus. An exception to this is the case where the ellipse is a circle. In this case by rotating it we just get the circle back. But his is no contradiction to the Liouville-Arnold theorem. Namely at the circle the Hamiltonian vector fields of E and L are parallel to each other, so that a circle does not lie on a regular value of the map $(E, L) : T^*(\mathbb{R}^2 \backslash \{0\}) \to \mathbb{R}^2$. For collision orbits the level set is non-compact so that we cannot apply the Liouville-Arnold theorem directly, either. Some special considerations need to be taken to avoid this annoying non-compactness. Usually that goal is achieved through a regularization technique, but we will not provide one here.

Apart from the physical symmetry obtained by rotation which is generated by the Hamiltonian vector field of angular momentum, the Kepler problem admits as well some "hidden symmetries". These hidden symmetries do not arise from flows on the configuration space $\mathbb{R}^2\backslash\{0\}$ but from flows which only live on phase space $T^*(\mathbb{R}^2\backslash\{0\})$. We discussed the Runge Lenz vector in Section 5.1.2. In the planar case the third component of this vector vanishes so that just the first two are of interest. They are the smooth functions

$$A_1, A_2 : T^*(\mathbb{R}^2\backslash\{0\}) \to \mathbb{R}$$

given by

$$\begin{cases} A_1(q,p) = p_2(p_2q_1 - p_1q_2) - \frac{q_1}{|q|} = p_2L(q,p) - \frac{q_1}{|q|}, \\ A_2(q,p) = -p_1(p_2q_1 - p_1q_2) - \frac{q_2}{|q|} = -p_1L(q,p) - \frac{q_2}{|q|}. \end{cases}$$

By Lemma 5.6 the Poisson bracket of E with A_1 and A_2 vanishes, i.e.,

$$\{E, A_1\} = \{E, A_2\} = 0.$$

For the planar Kepler problem the two dimensional vector

$$A = (A_1, A_2) : T^*(\mathbb{R}^2\backslash\{0\}) \to \mathbb{R} \tag{5.7}$$

is referred to as the **Runge-Lenz vector**. By lemma 5.7 we have the equality

$$A^2 = 1 + 2EL^2. \tag{5.8}$$

from which the following inequality for energy and angular momentum in the Kepler problem follows

$$2EL^2 + 1 \geq 0. \tag{5.9}$$

As we have seen in Section 5.1 the length of the Runge-Lenz vector A corresponds to the eccentricity of the conic section. Therefore equality holds if and only if the trajectory lies on a circle.

We finally work out the Hamiltonian of the planar Kepler problem in polar coordinates and deduce Kepler's second law. In polar coordinates for the q_xq_y-plane, there is a nice expression of L. We write $(q_x, q_y) = (r\cos\phi, r\sin\phi)$. The momentum coordinates (p_x, p_y) transform with the inverse of the Jacobian, so if we denote the cotangent coordinates dual to (r, ϕ) by (p_r, p_ϕ), then we find

$$(p_x, p_y) = (\cos\phi \cdot p_r - \frac{\sin\phi}{r}p_\phi, \sin\phi \cdot p_r + \frac{\cos\phi}{r}p_\phi).$$

The coordinate change for the q-coordinates is

$$q_x = q_r\cos(q_\phi), \quad q_y = q_r\sin(q_\phi).$$

We are looking for a symplectic transformation, so, as mentioned, we just need the inverse and transpose of the Jacobian of this coordinate transformation for the p-part. It is, however, convenient to compute by using the fact that the canonical 1-form, $\lambda = p_x\,dq_x + p_y\,dq_y$ is preserved. This gives the equation $p_x\,dq_x + p_y\,dq_y = p_r\,dq_r + p_\phi\,dq_\phi$, so we find $p_x = p_r\cos q_\phi - \frac{p_\phi}{q_r}$ and $p_y = p_r\sin q_\phi + \frac{p_\phi}{q_r}\cos q_\phi$. The Hamiltonian in polar coordinates is hence

$$E = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{q_r^2}\right) - \frac{1}{q_r}.$$

Remark 5.11. It is important to observe that p_ϕ is the angular momentum. The Hamiltonian equations clearly show that the angular momentum is preserved as we expect for a central force.

Plug this into $L = q^1p_2 - q^2p_1$ and use the Hamilton equations $\dot{q}^i = p_i$ to find.

Lemma 5.12. *(**Kepler's second law**) We have*

$$\frac{1}{2}r^2\dot{\phi} = \frac{1}{2}l = \frac{dArea}{dt}.$$

where Area is the area swept out by an ellipse.

To get the claim about the area, just use that

$$A = \int_{\phi=\phi_1}^{\phi_2} \int_{r=0}^{r=r_1} r\,dr\,d\phi = \int_{\phi=\phi_1}^{\phi_2} \frac{1}{2}r^2 d\phi.$$

We assume that $L \neq 0$. The case $L = 0$ can be worked out separately: it involves collision orbits.

Further reading

Chapter 4. of Frauenfelder and van Koert [8] is about a topic we already introduced: regularization techniques to avoid the problems raised by a possible collision of the two bodies.

The whole Chapter 9. of Abraham and Marsden [1] is devoted to the two body problem. One will find there different models and different sets of variables with which Kepler's problem can be approached.

Chapter 6

The restricted three body problem

The first ingredient in the restricted three body problem are two masses, the **primaries**, which we refer to as the Earth and the Moon. We scale the total mass to one so that for some $\mu \in [0,1]$ the mass of the moon equals μ and the mass of the earth equals $1-\mu$. Here we allow the mass of the moon to be bigger than the mass of the earth, although in such a situation one might prefer to change the names of the primaries. The earth and the moon move in 3-dimensional Euclidean space $\mathbb{R}^3$ according to Newton's law of gravitation and we denote their time dependent positions by $e(t) \in \mathbb{R}^3$ respectively $m(t) \in \mathbb{R}^3$ for $t \in \mathbb{R}$.

The second ingredient is a massless object referred to as the satellite. Because the satellite is massless it does not influence the movements of the earth and the moon. On the other hand the earth and the moon attract the satellite according to Newton's law of gravitation. The goal of the problem is to get an understanding of the dynamics of the satellite which can be quite intricate.

6.1 The restricted planar three body problem in an inertial frame

If q denotes the position of the satellite and p its momentum then the Hamiltonian of the satellite in the inertial system is given according to Newton's law of gravitation by

$$E_t(q,p) = \frac{1}{2}p^2 - \frac{\mu}{|q-m(t)|} - \frac{1-\mu}{|q-e(t)|} \tag{6.1}$$

namely the sum of kinetic energy and Newton's potential. We abbreviate this Hamiltonian by E and not by H in order to distinguish it from the Hamiltonian of the restricted three body problem in rotating coordinates. Note that because the earth and the moon are moving the Hamiltonian is no autonomous, i.e., it depends on time. Actually, because we have to avoid collisions of the satellite with one of the primaries even the domain of definition of the Hamiltonian is time dependent, namely

$$E_t : T^*(\mathbb{R}^2 \backslash \{e(t), m(t)\}) \to \mathbb{R}$$

Since we chose $n=2$, this is referred to as the **planar restricted three body problem**, while the former one is called the **spatial restricted three body problem**. In the following we focus on the planar case. This is due to the fact that the question about global surfaces of section only makes sense in the planar case. A further specialization is obtained by assuming that the earth and moon move on circles about their common center of mass. After choosing suitable coordinates their time dependent positions are given by

$$e(t) = -\mu(\cos(t), -\sin(t)), \quad m(t) = (1-\mu)(\cos(t), -\sin(t)). \tag{6.2}$$

This problem is referred to as the **circular planar restricted three body problem**. Of course there is also a circular spatial restricted three body problem. The amazing thing about the circular case is that after a time dependent transformation which puts the earth and moon at rest, the Hamiltonian of the circular restricted three body problem in rotating coordinates becomes autonomous, i.e., independent of time. In particular, it is preserved along its flow. This surprising observation is due to Jacobi. We first explain time dependent transformations.

6.2 Time dependent transformations

Suppose that (M, ω) is a symplectic manifold and $E \in C^\infty(M \times \mathbb{R})$ and $L \in C^\infty(M \times \mathbb{R})$ are two time dependent Hamiltonians. For $t \in \mathbb{R}$ abbreviate $E_t = E(\cdot, t) \in C^\infty(M)$ and similarly L_t. This gives rise to two time dependent Hamiltonian vector fields X_{E_t} and X_{L_t}. For simplicity let us assume that the flows of the Hamiltonian vector fields ϕ_E^t and ϕ_L^t exist for all times. One can consider more complicated situations where the domain of definitions of the two Hamiltonians itself depends on time. This actually happens in the restricted three body problem. Nevertheless the treatment of this more general case does not require basic new ingredients apart from a notational nightmare.

Define the time dependent Hamiltonian function

$$L \diamondsuit E \in C^\infty(M \times \mathbb{R})$$

by

$$(L \diamondsuit E)(x, t) = L(x, t) + E((\phi_L^t)^{-1} x, t), \quad x \in M,\ t \in \mathbb{R}.$$

We claim that

$$\phi_{L \diamondsuit E}^t = \phi_L^t \circ \phi_E^t, \quad t \in \mathbb{R}. \tag{6.3}$$

To see that pick $x \in M$. Abbreviate $y = \Phi_L^t(\phi_E^t(x))$ and pick further $\xi \in T_y M$. We compute using the fact that ϕ_E^t is symplectic from **Theorem 1.9**

$$\begin{aligned}
\omega\left(\frac{d}{dt}(\phi_L^t(\phi_E^t(x)), \xi\right) &= \omega\left(X_{L_t}(y) + d\phi_L^t(\phi_E^t(x)) X_{E_t}(\phi_E^t(x)), \xi\right) \\
&= dL_t(y)\xi + \omega\left(X_{E_t}((\phi_L^t)^{-1}(y)), (d\phi_L^t)^{-1}(y)\xi\right) \\
&= dL_t(y)\xi + d(E \circ (\phi_L^t)^{-1})(y)\xi \\
&= d(L \diamondsuit E)_t(y)_\xi.
\end{aligned}$$

This establishes (6.3).

Note that even if E and L are autonomous, i.e., independent of time, the Hamiltonian $L \diamondsuit E$ does not need to be autonomous, unless E is invariant under the flow of L.

6.3 The circular problem in a rotating frame

For simplicity we discuss the planar case. The spatial case works analogously. We apply to the Hamiltonian E_t given by (6.1) with positions of the earth and the moon determined by (6.2) the time dependent transformation generated by angular momentum

$$L \in C^\infty(T^*\mathbb{R}^2), \quad (q, p) \mapsto q_1 p_2 - q_2 p_1.$$

We abbreviate

$$H := L \diamondsuit E.$$

It is known that angular momentum generates the rotation. If we abbreviate

$$e = (-\mu, 0), \quad m = (1-\mu, 0)$$

the Hamiltonian H becomes

$$H(q,p) = \frac{1}{2}p^2 - \frac{\mu}{|q-m|} - \frac{1-\mu}{|q-e|} + q_1 p_2 - q_2 p_1. \tag{6.4}$$

Note that this Hamiltonian is autonomous. In particular, in the rotating frame the Hamiltonian H is preserved by **Theorem 1.8**. This surprising observation goes back to Jacobi and therefore H is also referred to as the **Jacobi integral**. [1]

We point out that the fact that $H = L\diamondsuit E$ is autonomous only holds in the circular case. For example if the primaries move on ellipses with some positive eccentricity, the so called elliptic restriced three body problem, the Hamiltonian H does not become time independent.

Abbreviating by

$$V : \mathbb{R}^2 \backslash \{e, m\} \to \mathbb{R}, \quad q \mapsto -\frac{\mu}{|q-m|} - \frac{1-\mu}{|q-e|}$$

the Newtonian potential the Hamiltonian equations of motion become

$$\begin{cases} q_1' = p_1 - q_2 \\ q_2' = -p_2 + q_1 \\ p_1' = -p_2 - \frac{\partial V}{\partial q_1} \\ p_2' = p_1 - \frac{\partial V}{\partial q_2}. \end{cases} \tag{6.5}$$

For the second derivatives of q we compute

$$q_1'' = p_1' - q_2' = -p_2 - \frac{\partial V}{\partial q_1} - p_2 - q_1 = -2q_2' + q_1 - \frac{\partial V}{\partial q_1}$$

and

$$q_2'' = p_2' + q_1' = p_1 - \frac{\partial V}{\partial q_2} + p_1 - q_2 = 2q_1' + q_2 - \frac{\partial V}{\partial q_2}.$$

Therefore the first order ODE (6.5) is equivalent to the following second order ODE

$$\begin{cases} q_1'' = -2q_2' + q_1 - \frac{\partial V}{\partial q_1} \\ q_2'' = 2q_1' + q_2 - \frac{\partial V}{\partial q_2}. \end{cases} \tag{6.6}$$

To give the additional rotational term a physical interpretation we complete the squares and rewrite (6.4) as

$$H(q,p) = \frac{1}{2}\Big((p_1 - q_2)^2 + (p_2 + q_1)^2\Big) - \frac{\mu}{|q-m|} - \frac{1-\mu}{|q-e|} - \frac{1}{2}q^2. \tag{6.7}$$

The last three terms only depend on q and we introduce the so called **effective potential**

$$U : \mathbb{R}^2 \backslash \{e, m\} \to \mathbb{R}, \quad q \mapsto -\frac{\mu}{|q-m|} - \frac{1-\mu}{|q-e|} - \frac{1}{2}q^2 = V(q) - \frac{1}{2}q^2.$$

Using this abbreviation the Hamiltonian H can be written more compactly as

$$H(q,p) = \frac{1}{2}\Big((p_1 - q_2)^2 + (p_2 + q_1)^2\Big) + U(q). \tag{6.8}$$

[1] More precisely, for some historic reasons the integral $-2H$, which of course is preserved under the Hamiltonian flow of H as well, is traditionally called the Jacobi integral.

The effective potential consists of the Newtonian potential for the earth and the moon plus the additional term $-\frac{1}{2}q^2$. The additional term gives rise to a new force just experienced in rotating coordinates, namely the **centrifugal force**. The Hamiltonian H in (6.8) is not a mechanical Hamiltonian anymore, i.e., it does not just consist of kinetic plus potential energy. Instead of that the Hamiltonian contains a twist in the kinetic part and is therefore a magnetic Hamiltonian as discussed in Section 5.1.2. The twist in the kinetic part can be interpreted in terms of physics as an additional force, namely the **Coriolis force**. Different from the gravitational force and the centrifugal force which only depend on the position of the satellite the Coriolis force depends on its velocity, like the Lorentz force for a particle moving in a magnetic field. There are now four forces acting on the satellite in the rotating coordinate system, the gravitational force of the earth, the gravitational force of the moon, the centrifugal force, as well as the Coriolis force. This vividly shows that the dynamics of the restricted three body complex is highly intricate.

Further reading

The set of titles that continue from somewhere near of where we left this topic is arguably finite. Just as an example, Chapter 10. of Abraham and Marsden [1] is one of the main treaties of its time.

Arnold's [2] 8*th* appendix presents the theory of perturbations of conditionally periodic motion and Kolmogorov's theorem, which are so important on today's concerns with the restricted three body problem.

Also from Chapter 5. on, Frauenfelder and van Koert [8] proposes the use of holonomic curves to study the restricted three body problem.

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